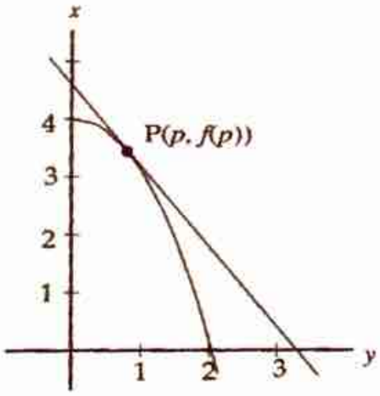
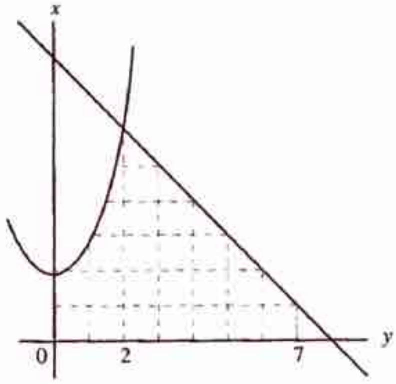
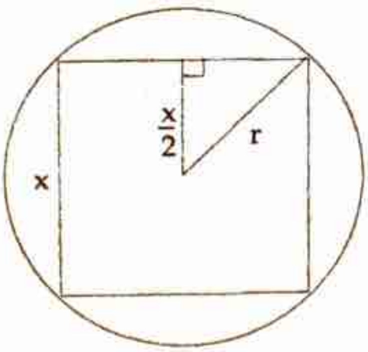
Complete Solutions Manual to Accompany

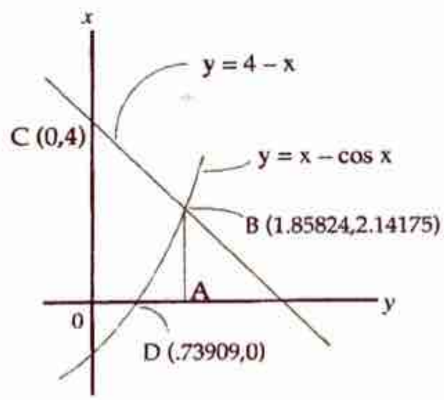
PREPARING FOR THE

AP CALCULUS (AB) EXAMINATION









David Penner
Richard Lux

Table of Contents

EXAM I
Section I Part A 1
Section I Part B 8
Section II Part A 13
Section II Part B 16

EXAM II
Section I Part A 19
Section I Part B 26
Section II Part A 31
Section II Part B 33

EXAM III
Section I Part A 35
Section I Part B 41
Section II Part A 46
Section II Part B 49

EXAM IV
Section I Part A 51
Section I Part B 57
Section II Part A 61
Section II Part B 63

EXAM V
Section I Part A 66
Section I Part B 72
Section II Part A 76
Section II Part B 78

EXAM VI
Section I Part A 81
Section I Part B 88
Section II Part A 93
Section II Part B 96

Exam I
Section I
Part A — No Calculators

1. A p. 1

$f(x) = x^2 + e^{\sin x}$

$f'(x) = 2x + e^{\sin x} \cdot \cos x$

$f'(0) = 1$

Snce $f'(0) > 0$, f is increasing. The answer is A.

2. B p. 1

$y = x \cos x$

$\frac{dy}{dx} = \cos x - x \sin x$

$\cos x - x \sin x = 0$

$\cos x = x \sin x$

$\frac{1}{x} = \tan x$

3. E p. 2

With $F(x) = G[x + G(x)]$, the Chain Rule gives

$F'(x) = G'[x + G(x)] \cdot (1 + G'(x))$

Then $F'(1) = G'[1 + G(1)] \cdot (1 + G'(1))$

From the graph of the function G, we find $G(1) = 3$.

Hence $F'(1) = G'[1 + 3] \cdot (1 + G'(1))$
$= G'(4) \cdot (1 + G'(1))$

From the graph of G we can determine that $G'(4) = \frac{2}{3}$ and $G'(1) = -2$.

Thus, $F'(1) = \frac{2}{3} \cdot (-1) = -\frac{2}{3}$.

4. E p. 2

$$\int_0^{\pi/4} \cos 2x \, dx = \frac{1}{2} \int_0^{\pi/4} \cos 2x \cdot 2 \, dx = \frac{1}{2}\left[\sin 2x\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\sin \frac{\pi}{2}\right) - \left(\sin 0\right)\right] = \frac{1}{2}[1 - 0] = \frac{1}{2}.$$

The answer is E.

5. E p. 2

$$f(x) = \frac{(\ln x)^2}{x}$$

$$f'(x) = \frac{x \cdot 2(\ln x) \cdot \frac{1}{x} - (\ln x)^2}{x^2} = \frac{(\ln x) \cdot (2 - \ln x)}{x^2}$$

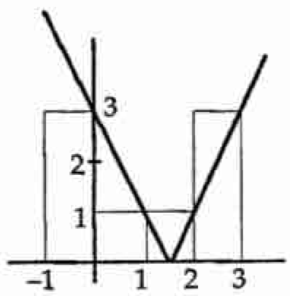
The critical numbers are $x = 1$ and $x = e^2$.

$x > e^2$	$\Rightarrow$	$f'(x) < 0$	$\Rightarrow$	f is decreasing.
$1 < x < e^2$	$\Rightarrow$	$f'(x) > 0$	$\Rightarrow$	f is increasing.
$0 < x < 1$	$\Rightarrow$	$f'(x) < 0$	$\Rightarrow$	f is decreasing.

The relative maximum is at $x = e^2$.

6. D p. 3

Graph the function $f(x) = |2x - 3|$ on the interval $[-1,3]$. Since the interval has length 4 and the Riemann sum is to have 4 equal subdivisions, each subdivision has length 1. Since it is to be a right-hand Riemann sum, we use function values at the right-hand ends of the intervals; that is, at $x = 0, 1, 2$, and 3.



$$R_4 = 1 \cdot [f(0) + f(1) + f(2) + f(3)]$$
$$= 1 \cdot [3 + 1 + 1 + 3] = 8$$

7. A p. 3

$$y = x^3 + 3x^2 + 2 \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2 + 6x$$

$$\frac{d^2y}{dx^2} = 6x + 6 = 0 \quad \Rightarrow \quad x = -1 \quad \Rightarrow \quad \begin{cases} y = 4 \\ \frac{dy}{dx} = -3 \end{cases}$$

Hence the point of inflection is $(-1,4)$ and the slope of the tangent is -3.
Then the equation of the tangent is $y - 4 = -3(x + 1)$, so $y = -3x + 1$.

8. D p. 3

$$\int \cos(3 - 2x)\,dx = -\frac{1}{2} \int (-2)\cos(3 - 2x)\,dx$$
$$= -\frac{1}{2} \sin(3 - 2x) + C$$

9. B p. 4

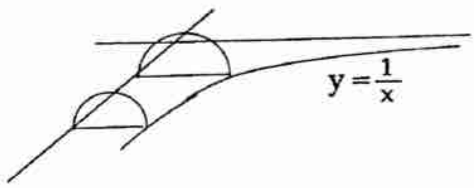
$$\lim_{x\to\infty}\frac{\sqrt{9x^2+2}}{4x+3}=\lim_{x\to\infty}\frac{x\sqrt{9+\frac{2}{x^2}}}{4x+3}=\lim_{x\to\infty}\frac{\sqrt{9+\frac{2}{x^2}}}{4+\frac{3}{x}}=\frac{3}{4}$$

---

10. B p. 4

Each cross section perpendicular to the x-axis (at coordinate x) is a semicircle of radius $\frac{1}{2x}$. The cross-sectional area is $\frac{1}{2}\pi\left(\frac{1}{2x}\right)^2$.

Hence the volume of the solid is given by:



$$V=\int_1^4 \frac{1}{2}\pi\left(\frac{1}{2x}\right)^2 dx=\frac{\pi}{8}\int_1^4\frac{1}{x^2}\,dx=\frac{\pi}{8}\left[-\frac{1}{x}\right]_1^4=\frac{\pi}{8}\left(-\frac{1}{4}+1\right)=\frac{3\pi}{32}.$$

---

11. B p. 4

$$g(x)=\int_{-2}^{x}f(t)\,dt \;\Rightarrow\; g'(x)=f(x) \;\Rightarrow\; g''(x)=f'(x)$$

The slope of the graph of f at $x=0$ is $\frac{4}{3}$.

Since the derivative of f at $x=0$ is the slope of the graph of f at $x=0$, we have $g''(0)=f'(0)=\frac{4}{3}$.

---

12. D p. 5

$$F(x)=\int_0^x\frac{2}{1+t^4}\,dt \text{ thus } F'(x)=\frac{2}{1+x^4} \text{ and } F''(x)=\frac{0-8x^3}{(1+x^4)^2}=\frac{-8x^3}{(1+x^4)^2}$$

I. $F(0)=\int_0^0\frac{2}{1+t^4}\,dt=0$. False

II. $F(2)=\int_0^2\frac{2}{1+t^4}\,dt$ and $F(6)=\int_0^6\frac{2}{1+t^4}\,dt=\int_0^2\frac{2}{1+t^4}\,dt+\int_2^6\frac{2}{1+t^4}\,dt$

Since $\frac{2}{1+x^4}$ is a positive valued area function, then $F(2)<F(6)$. True

III. $F''(0)=\frac{0}{(1+0^4)^2}=0$ True

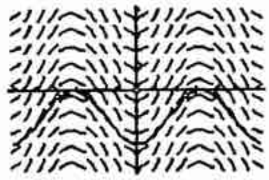
---

13. C p. 5

$$\frac{1}{1-(-1)}\int_{-1}^{1}(2t^3-3t^2+4)\,dt = \frac{1}{2}\left[\frac{t^4}{2}-t^3+4t\right]_{-1}^{1}$$
$$= \frac{1}{2}\left[\left(\frac{1}{2}-1+4\right)-\left(\frac{1}{2}+1-4\right)\right] = \frac{1}{2}\cdot 6 = 3$$

14. B p. 5

Draw a solution curve on the slope field. This looks like an up-side down cosine curve. That is, the solution of the differential equation for which we have a slope field is $y = -\cos x$.
The differential equation is $\frac{dy}{dx} = \sin x$.



15. B p. 6

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)}$$
$$= \lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{1}{(\sqrt{x}+1)} = \frac{1}{2}$$

16. E p. 6

$$y = \cos^2 x - \sin^2 x$$
$$y' = -2\cos x\sin x - 2\sin x\cos x = -4\sin x\cos x$$

17. C p. 6

$$\int_1^2 \left(4x^3+6x-\frac{1}{x}\right)dx = \left(x^4+3x^2-\ln|x|\right]_1^2$$
$$= (16+12-\ln 2)-(1+3-\ln 1) = (24-\ln 2)$$

18. C p. 7

$$\int\frac{x-2}{x-1}\,dx = \int\frac{(x-1)-1}{x-1}\,dx = \int\left[1-\frac{1}{x-1}\right]dx = x-\ln|x-1| + C$$

19. B p. 7

The property that $g(-x) = g(x)$ for all x means that the function g is even. Its symmetry around the y-axis guarantees that $g'(-a) = -g'(a)$.

More formally, differentiating the first property gives

$$g'(-x) \cdot (-1) = g'(x).$$

Thus $$g'(-x) = -g'(x).$$

---

20. A p. 7

$y = \text{Arctan}\,\frac{x}{3}$

$y' = \frac{1}{3}\,\frac{1}{1+\frac{x^2}{9}} = \frac{3}{9+x^2}$. This implies that $y'(0) = \frac{1}{3}$.

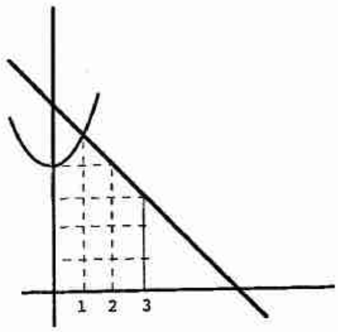
Hence the line goes through the origin with slope $\frac{1}{3}$.

Its equation is $y - 0 = \frac{1}{3}(x-0)$, which can be written $x - 3y = 0$.

---

21. C p. 8

Solution I. With a reasonably careful graph, it is possible to obtain an estimate of the definite integral by counting the squares under the graph of $f(x)$ on the interval $[0,3]$.



Solution II. Having determined that the change in the function definition occurs at $x = 1$, evaluate $\int_0^3 f(x)\,dx$.

This is done in two parts, as:

$$\int_0^3 f(x)\,dx = \int_0^1 (x^2+4)\,dx + \int_1^3 (6-x)\,dx$$

$$= \left[\frac{x^3}{3} + 4x\right]_0^1 + \left[6x - \frac{x^2}{2}\right]_1^3$$

$$= \left[\frac{1}{3} + 4\right] - 0 + \left[18 - \frac{9}{2}\right] - (6 - \frac{1}{2}) = 12 + \frac{1}{3}$$

---

22. E p. 8

$f(x) = \ln(e^{2x} + 3\sin x)$

$f'(x) = \frac{1}{e^{2x}+3\sin x}(2e^{2x} + 3\cos x)$

Tangent slope is $f'(0) = \frac{1}{e^0+3\sin 0}(2e^0 + 3\cos 0) = \frac{1}{1+0}(2+3\cdot 1) = 5$. The answer is E

---

23. D p. 8

$g'(x) = 2g(x) \Rightarrow \frac{g'(x)}{g(x)} = 2$

Integrating gives $\ln|g(x)| = 2x + C$

Then $g(x) = e^{2x+C}$

Using the initial condition that $g(-1) = 1$, we have

$$g(-1) = e^{-2+C} = 1 \Rightarrow C = 2$$

Hence $g(x) = e^{2x+2}$

The notation is simpler if we let $y = g(x)$. Then the equation is $y' = 2y$. The solution proceeds as before.

$y' = 2y \Rightarrow \frac{y'}{y} = 2 \Rightarrow \ln|y| = 2x + C$

$\Rightarrow y = \pm e^{2x+C}$ Since $g(-1) = 1$, $y = e^{2x+C}$

---

24. D p. 9

We antidifferentiate the acceleration function to obtain the velocity.

$a(t) = 3t + 2 \Rightarrow v(t) = \frac{3}{2}t^2 + 2t + C$

$v(1) = 4 \Rightarrow 4 = \frac{3}{2} + 2 + C \Rightarrow C = \frac{1}{2}$

Thus $v(t) = \frac{3}{2}t^2 + 2t + \frac{1}{2}$

Antidifferentiate again to obtain $x(t) = \frac{1}{2}t^3 + t^2 + \frac{1}{2}t + D$

$x(1) = 6 \Rightarrow 6 = \frac{1}{2} + 1 + \frac{1}{2} + D \Rightarrow D = 4$

Then the position function is $x(t) = \frac{1}{2}t^3 + t^2 + \frac{1}{2}t + 4$.

Hence $x(2) = 4 + 4 + 1 + 4 = 13$.

---

25. B p. 9

$y = \sqrt{3 + e^x}$ passes through $(0,2)$.

$\frac{dy}{dx} = \frac{e^x}{2\sqrt{3+e^x}}$; when $x = 0$, this has a value of $\frac{1}{4}$.

The equation of the tangent line at $(0,2)$ is $y - 2 = \frac{1}{4}x$, or $y = 2 + \frac{1}{4}x$.

When $x = 0.08$, $y = 2 + \frac{1}{4}(.08) = 2.02$.

26. B p. 9

For $1 < t < 3$, the leaf rises 5 feet in 2 seconds. $s = \frac{5}{2} = 2.5$ ft/sec.

For $3 < t < 5$, the leaf falls 10 feet in 2 seconds. $s = \frac{10}{2} = 5$ ft/sec.

For $5 < t < 7$, the leaf rises 3 feet in 2 seconds. $s = \frac{3}{2} = 1.5$ ft/sec.

For $7 < t < 9$, the leaf falls 8 feet in 2 seconds. $s = \frac{8}{2} = 4$ ft/sec.

Since the slope of the graph is constant on each of these intervals, the only other interval of interest is $0 < t < 1$. During that period, the leaf rises 1.5 feet in 1 second. Then $s = \frac{1.5}{1} = 1$ ft/sec.

The maximum speed is 5 ft/sec, occurring in the interval $3 < t < 5$.

---

27. C p. 10

Differentiating the given volume function with respect to t gives

$$\frac{dV}{dt} = \pi(12h - h^2)\frac{dh}{dt}.$$

We know $\frac{dV}{dt} = 30\pi$ ft^3/sec, and are interested in $\frac{dh}{dt}$ when $h = 2$ ft. Substituting these values, we have

$30\pi = \pi(12 \cdot 2 - 2^2)\frac{dh}{dt}$. Hence $\frac{dh}{dt} = \frac{30\pi}{20\pi} = 1.5$ ft/hr.

---

28. E p. 10

$f(x) = 2x^{5/3} - 5x^{2/3} \Rightarrow f'(x) = \frac{10}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{10}{3}x^{-1/3}(x-1)$

The function f has two critical numbers:

$x = 1$ (where $f'(x) = 0$) and $x = 0$ (where $f'(x)$ is undefined).

To determine the sign of the first derivative, we consider the intervals into which these critical numbers divide the domain of the function.

	$x<0$	$0<x<1$	$x>1$
$x^{-1/3}$	−	+	+
$x-1$	−	−	+
$f'(x)$	+	−	+

$f(x)$ is increasing if and only if $f'(x) > 0$. This occurs if $x < 0$ or $x > 1$.

Exam I
Section I
Part B — Calculators Permitted

1. D p. 11

I. $\lim_{x \to 1} f(x) = -1$ — False

II. $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h} = f'(2) = 2$ — True

III. $\lim_{x \to -1^+} f(x) = 1 = f(-3)$ — True

2. C p. 11

$f(x) = \sin^2 x \quad \Rightarrow \quad f'(x) = 2\sin x \cos x = \sin(2x)$

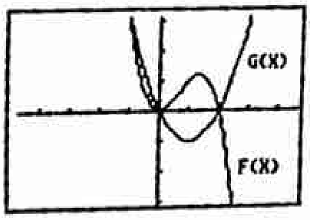
$g(x) = .5x^2 \quad \Rightarrow \quad g'(x) = x$

From a calculator graph of the functions f' and g', we see the only possible solution is $x = 0.9$.

3. B p. 12

Here are two possible calculator solutions.

I. First, look at graphs of the functions f and g, and find those intervals where the graph of f is above the graph of g.



The cubic function $f(x)$ is above the quadratic function $g(x)$ when x is between 0 and 2. Thus, the integral of f will have a larger value than the integral of g on the intervals [0,2].

II. Second, use the calculator to evaluate these definite integrals on the intervals [a,b] indicated.

		$\int_a^b f(x)\,dx$	$\int_a^b g(x)\,dx$	
I. a = −1	b = 0	.917	1.333	False
II. a = 0	b = 2	1.333	−1.333	True
III. a = 2	b = 3	−3.583	1.333	False

4. E p. 12

$y^2 - 3x = 7 \quad \Rightarrow \quad 2y\frac{dy}{dx} - 3 = 0$

$\frac{dy}{dx} = \frac{3}{2y}$

$\frac{d^2y}{dx^2} = \frac{2y \cdot 0 - 3 \cdot 2\frac{dy}{dx}}{4y^2} = \frac{-6 \cdot \frac{3}{2y}}{4y^2} = -\frac{9}{4y^3}$

---

5. D p. 12

I. $h(0) = g(f(0)) = g(5) = 0$ — False

II. $h'(x) = g'(f(x)) \cdot f'(x)$

Thus $h'(2) = g'(f(2)) \cdot f'(2)$

$= g'(1) \cdot (-\frac{1}{4}) = (-2) \cdot (-\frac{1}{4}) > 0$ — True

III. $h'(4) = g'(f(4)) \cdot f'(4) = g'(2) \cdot 1 = 0 \cdot 1 = 0$ — True

---

6. D p. 13

If (x, e^x) is on the curve, then its distance from the origin is

$D = \sqrt{x^2 + e^{2x}}$. Use a calculator graph of this distance function and find its minimum.

At $x = -0.426$, the minimum distance of 0.78 is achieved.

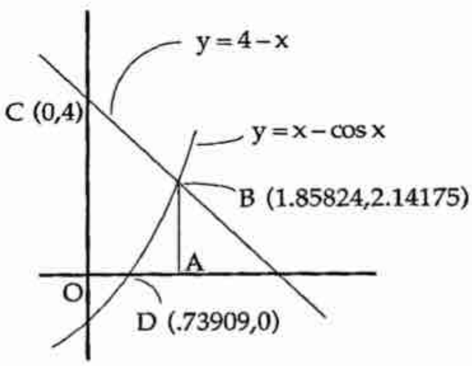
---

7. B p. 13

In the figure, we need the area of the region ODBC. This can be calculated as area of trapezoid OABC − area DAB. The coordinates of points B and D are found using the calculator. Then the desired area is

$$a = \int_0^{1.858} (4 - x)\,dx - \int_{.739}^{1.858} (x - \cos x)\,dx$$

$\approx 4.54.$



---

8. B p. 13

$y = x^4 - x^2 - e^{2x}$

$y' = 4x^3 - 2x - 2e^{2x}$

$y'' = 12x^2 - 2 - 4e^{2x}$

Graph the second derivative on the interval $[-2, 2]$ and search for zeros. The graph shows that y" changes sign at $x = -0.531$.

9. C p. 14

$\frac{dV}{dt} = \sqrt{1+2^t} \Rightarrow V = \int_0^5 \sqrt{1+2^t}\,dt \approx 14.53$ ft^3.

10. C p. 14

We use the disk (washer) method. $V = \pi \int_1^6 [f(x)]^2\,dx$

Using the Trapezoid Rule with five subintervals to approximate this, we obtain

$$V \approx T_5 = \frac{\pi}{2}\left[f^2(1) + 2\cdot f^2(2) + 2\cdot f^2(3) + 2\cdot f^2(4) + 2\cdot f^2(5) + f^2(6)\right]$$
$$= \frac{\pi}{2}\left[2^2 + 2\cdot 3^2 + 2\cdot 4^2 + 2\cdot 3^2 + 2\cdot 2^2 + 1^2\right]$$
$$= \frac{\pi}{2}\cdot 81 \approx 127$$

11. E p. 14

Solution I. We can do the problem algebraically:
Given the position function $x(t) = (t+1)(t-3)^3$,
we differentiate to obtain the velocity function:

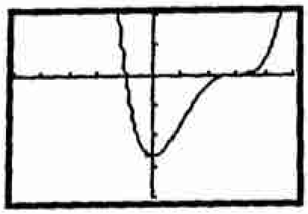
$$v(t) = (t+1)\cdot 3(t-3)^2 + (t-3)^3 = 4t\,(t-3)^2$$

For the velocity to be increasing, we need $v'(t) > 0$.

$$v'(t) = (t-3)^2\cdot 4 + (4t)\cdot 2(t-3) = 12(t-3)(t-1).$$

We find that $v'(t) > 0$ if $t > 3$ or $t < 1$.

Solution II. Alternatively, we can do the problem graphically. Given the position $x(t)$, the velocity is $v(t) = x'(t)$. For the velocity to be increasing, we need $v'(t) > 0$. That is to say, we need $x''(t) > 0$; hence we want the graph of $x(t)$ to be concave up. From the graph of $x(t)$ shown, we recognize that the curve is concave up when $x < 1$ and again when $x > 3$.



12. E p. 15

$$f(x) = \frac{\ln e^{2x}}{x-1} = \frac{2x}{x-1}.$$

The inverse of this function is found by solving $x = \frac{2y}{y-1}$ for y.

$$x = \frac{2y}{y-1} \Rightarrow xy - x = 2y \Rightarrow xy - 2y = x \Rightarrow y(x-2) = x$$
$$\Rightarrow g(x) = y = \frac{x}{x-2}$$

Then $g'(x) = \frac{(x-2)-x}{(x-2)^2} = \frac{-2}{(x-2)^2}$. Hence $g'(3) = -2$.

13. C p. 15

Divide the integrand fraction and rewrite the second term.

$$\int \frac{e^{x^2} - 2x}{e^{x^2}}\,dx = \int \left[1 - \frac{2x}{e^{x^2}}\right] dx = \int \left[1 - 2x\,e^{-x^2}\right] dx$$

$$= \int \left[1 + e^{-x^2}(-2x)\right] dx$$

In the second term of the integrand, the factor $(-2x)$ is the derivative of the exponent in the factor e^{-x^2}. Hence we can perform the antidifferentiation:

$$\int \left[1 + e^{-x^2}(-2x)\right] dx = x + e^{-x^2} + C.$$

---

14. C p. 16

$$f'(x) = \frac{(x-1)(x-4)^3}{1+x^4}$$

I. $f'(2) = \frac{(1)(-8)}{1+16} = -\frac{8}{17} \neq -8$ False

II. On $(1, 4)$, $x - 1 > 0$, $(x-4)^3 < 0$ and $1 + x^4 > 0$
Hence, $f'(x) < 0$ on $(1, 4)$ and F is decreasing there. False

III. $f'(3) < 0$ $f'(4) = 0$ $f'(5) > 0$ Hence, $f(4)$ is a relative minimum. True

---

15. B p. 16

For continuity, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \Rightarrow 3 + 3b = m + b$

For differentiability, $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x) \Rightarrow 3b + 4 = m$

We solve these two equations simultaneously:

$$\begin{cases} 2b + 3 = m \\ 3b + 4 = m \end{cases} \Rightarrow b = -1 \text{ and } m = 1.$$

---

16. D p. 17

Solution I. On each two-second time interval, we can approximate the speed by using the average of the speeds at the beginning and the end of the interval.

On the interval $[0,2]$, speed $\approx$ 33 ft/sec. Distance traveled $\approx$ 66 ft.
On the interval $[2,4]$, speed $\approx$ 38 ft/sec. Distance traveled $\approx$ 76 ft.
On the interval $[4,6]$, speed $\approx$ 44 ft/sec. Distance traveled $\approx$ 88 ft.
On the interval $[6,8]$, speed $\approx$ 51 ft/sec. Distance traveled $\approx$ 102 ft.
On the interval $[8,10]$, speed $\approx$ 57 ft/sec. Distance traveled $\approx$ 114 ft.
If we add these approximate distances traveled, we obtain 446 ft.

Solution II. Since $v(t) > 0$, on the interval [0, 10], the distance is the value of the integral $\int_0^{10} v(t)\,dt$.

Using Left and Right Riemann Sums, we approximate the integral as follows:

$$L_5 = 2[30 + 36 + 40 + 48 + 54] = 416$$

$$R_5 = 2[36 + 40 + 48 + 54 + 60] = 476$$

$$\text{Distance} = \int_0^{10} v(t)\,dt = \frac{L_5 + R_5}{2} = \frac{416 + 476}{2} = 446$$

17. E p. 17

Rewrite the given formula: $F(x) = -5 + \int_2^x \sin(\frac{\pi t}{4})\,dt$.

We obtain $F'(x)$ by using the Fundamental Theorem:

$$F'(x) = 0 + \sin(\frac{\pi x}{4}).$$

We can then evaluate both $F(2)$ and $F'(2)$.

$$F(2) = -5 + \int_2^2 \sin(\frac{\pi t}{4})\,dt = -5 + 0 = -5$$

$$F'(2) = \sin(\frac{2\pi}{4}) = \sin\frac{\pi}{2} = 1.$$

Then $F(2) + F'(2) = -5 + 1 = -4$.

Exam I
Section II
Part A — Calculators Permitted

1. p. 19

(a) Since the graph of f is a straight line here, $f'(3)$ = slope $= \dfrac{-3-3}{4-2} = -3$.

2: { 1: difference quotient; 1: answer }

(b) Using a linear approximation between $(-1,-1)$ and $(1,2)$, $g'(0) \approx \dfrac{2-(-1)}{1-(-1)} = \dfrac{3}{2}$.

2: { 1: difference quotient; 1: answer }

(c) $h(x) = g[f(x)]$

i) $h(2) = g[f(2)] = g(3) = 0$.

ii) $h'(x) = g'[f(x)]\cdot f'(x)$

$h'(3) = g'[f(3)]\cdot f'(3) = g'(0)\cdot(-3) = \dfrac{3}{2}(-3) = -\dfrac{9}{2}$.

3: { 1: $h(2)$; 2: $h'(3)$ }

(d) Using areas, we approximate $\int_0^4 f(x)\,dx$ as a trapezoid plus a triangle minus a triangle.

$$\int_0^4 f(x)\,dx = \tfrac{1}{2}(2)(1+3) + \tfrac{1}{2}(1)(3) - \tfrac{1}{2}(1)(3) \approx 4.$$

2: answer

2. p. 20

(a) To find the coordinates of Q, we write the equation of the tangent to the graph of $y = f(x)$ at the point $P(-2,8)$.

$f'(x) = 3x^2 + 6x - 1.$

Using $x = -2$, we find $f'(-2) = 12 - 12 - 1 = -1$. The line through the point $P(-2,8)$ with slope $m = -1$ is $y - 8 = -1(x+2)$ which can be rewritten: $y = -x + 6$.

3: { 1: slope; 1: tangent equation; 1: intersection pt }

We now solve simultaneously the equation of the cubic and the equation of the tangent line.

$$\begin{cases} y = x^3 + 3x^2 - x + 2 \\ y = -x + 6 \end{cases} \Rightarrow x^3 + 3x^2 - x + 2 = -x + 6$$

$$\Rightarrow x^3 + 3x^2 - 4 = 0$$

$$\Rightarrow (x+2)(x^2 + x - 2) = 0$$

$$\Rightarrow (x+2)(x+2)(x-1) = 0$$

There is the known intersection point where $x = -2$. The new point has an x-coordinate of $x = 1$. The corresponding y-coordinate is $y = 5$. Hence Q is the point (1,5).

(b) To find the inflection point R, we need $f''(x)$.

$f''(x) = 6x + 6.$

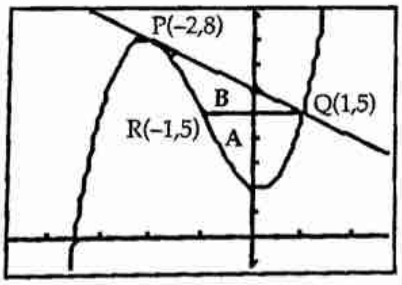
$f''(x) = 0$ if and only if $x = -1$.

When $x = -1$, the cubic function has a y-value of 5.

At $x = -1$, the value of $f''(x)$ changes from negative to zero to positive, hence the point of inflection of the graph of f occurs at R(−1, 5).

3: { 1: Solves $f''(x)=0$; 1: point; 1: justification }

(c) Shown to the right is a graph of the function f, with the points P(−2,8), Q(1,5), and R(−1,5) identified. To find the areas of the two regions described, we determine the area of the combined region $A \cup B$ and then determine the area of the larger of the two regions, region A.



3: { 1: area of region *A*; 1: area of region *B*; 1: ratio }

Area of region $A \cup B = \int_{-2}^{1} [(-x+6) - (x^3 + 3x^2 - x + 2\,dx)] = 6.75.$

Area of region $A = \int_{-1}^{1} [5 - (x^3 + 3x^2 - x + 2)]\,dx = 4.$

By subtraction, Area of region B = 2.75.

The ratio of these areas is $\frac{\text{Area of region A}}{\text{Area of region B}} = \frac{4}{2.75} = \frac{16}{11} = 1.455.$

Exam I
Section II
Part B — No Calculators

3. p. 21

(a) $\frac{dv}{dt} = \frac{v(30) - v(20)}{30 - 20} = \frac{11 - 7}{10} = 0.4 \text{ ft / sec}^2$

2: { 1: a difference quotient using numbers from table and interval contains 25
1: 0.4 ft / sec^2 }

(b) Midpoint sum = $20[v(10) + v(30) + v(50)] = 20(14 + 11 + 40) = 1300$ feet

2: { 1: uses $v(10), v(30), v(50)$
1: answer }

(c) $v_B(t) = \int \frac{1}{\sqrt{t+9}}\,dt = 2\sqrt{t+9} + C$

$3 = v(0) \Rightarrow 6 + C \Rightarrow C = -3$

$v_B(t) = 2\sqrt{t+9} - 3$

$v_B(40) = 14 - 3 = 11 < 12 = v_A(40)$

Car A is traveling faster at time $t = 40$ seconds,

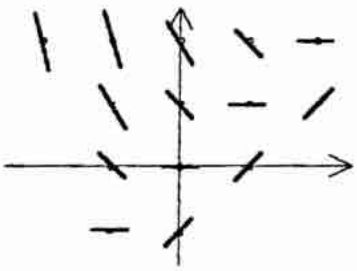
5: { 1: $2\sqrt{t+9}$
1: constant
1: use initial conditions
1: finds $v_B(40)$
1: compares to $v_A(40)$ and draws conclusion }

4. p. 22

(a) We calculate slopes at each of the fourteen points.

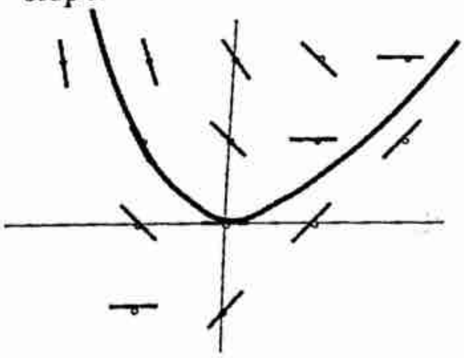
At (–2,2), m = –4.	At (–1,2), m = –3.
At (0,2), m = –2.	At (1,2), m = –1.
At (2,2), m = 0.	At (–1,1), m = –2.
At (0,1), m = –1.	At (1,1), m = 0.
At (2,1), m = 1.	At (0,–1), m = 1.
At (0,0), m = 0.	At (0,1), m = –1.
At (–1,–1), m = 0.	At (0,–1), m = 1.

Then draw short line segments through each of the points with the appropriate slope.



2: { 1: zero slopes; 1: nonzero slopes }

(b)



2: { 1: go through (–1,1); 1: follow the given slope lines and extend to the boundary of the slope field. }

(c) At the point (1,0), $\frac{dy}{dx} = 1 - 0 = 1$. Hence the slope of the straight line solution must be $m = 1$. The line through the point (1,0) with slope $m = 1$ is $y - 0 = 1(x - 1)$.
Hence the solution is $y = x - 1$.

2: { 1: slope; 1: tangent equation }

(d) Given the function $y = x - 1 + Ce^{-x}$, we have $\frac{dy}{dx} = 1 - Ce^{-x}$.
We can also write the expression $x - y$ in terms of x:
$x - y = x - \left(x - 1 + Ce^{-x}\right)$.
This simplifies to $x - y = 1 - Ce^{-x}$.
Thus, if $y = x - 1 + Ce^{-x}$, then $\frac{dy}{dx} = x - y$.

3: { 1: $\frac{dy}{dx} = 1 - Ce^{-x}$; 1: substitution; 1: conclusion }

5. p. 23

(a) $f'(3) = 2$. Hence the slope of the tangent line at the point $(3,1)$ is $m = 2$. Then an equation of the tangent line (in point-slope form) is:

$$y - 1 = 2(x - 3).$$

2: { 1: slope; 1: tangent equation }

(b) f has critical values at the points where $x = 1$ and $x = -3$, because $f'(x) = 0$.
To the immediate left of $x = 1$, $f'(x) < 0$, implying f is decreasing there.
To the immediate right of $x = 1$, $f'(x) > 0$, implying f is increasing there.
Since f is decreasing to the left of $x = 1$ and increasing to the right of $x = 1$, there is a local minimum there. Both to the left and right of $x = -3$, $f'(x) < 0$, so there is no relative max/min there.

2: { 1: answer; 1: justification }

(c) $f''(2)$ is the slope of the graph of $f'(x)$ at $x = 2$. Draw an estimate for the tangent line to $f'(x)$ at $x = 2$. Pick two points, such as $(1.2, 1)$ and $(3, 2.5)$, and the slope is $\frac{2.5 - 1}{3 - 1.2} = \frac{1.5}{1.8} = \frac{5}{6}$. (Any answer between 0.5 and 1.25 would be satisfactory.)

1: answer

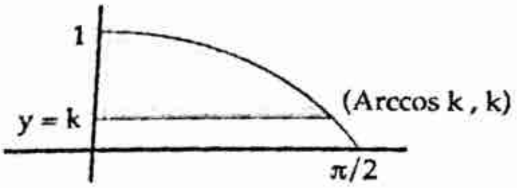
(d) The graph of f has inflection points at $x = -3, -1$ and 3 because the slope of f' changes sign at $x = -3, -1$ and 3. Put another way, the derivative of f has local extrema at $x = -3, -1$ and 3,

2: { 1: answer; 1: justification }

(e) The only candidates for maximum value are the endpoints $x = 0$ and $x = 4$, and the critical number at $x = 1$. In part (b) it was established that f has a local minimum at the $x = 1$. So the maximum value occurs at an endpoint. At $x = 0$, $f(0) = \int_0^0 f'(x)\,dx = 0$. Since the area of the region below the x-axis is smaller the the area of the region above the x-axis, $f(4) = \int_0^4 f'(x)\,dx > 0$. Hence f has its maximum value for that interval at the right-hand endpoint, $x = 4$.

2: { 1: answer; 1: Justification }

6. p. 24



(a)
$$\text{area} = \int_0^{\text{Arccos } k} (\cos x - k)\,dx = \left[\sin x - kx\right]_0^{\text{Arccos } k}$$

$$= \sin(\text{Arccos } k) - k \text{ Arccos } k$$

Note: Letting $A = \text{Arccos } k$, we have $\cos A = k$ and $\sin A = \sqrt{1-\cos^2 x} = \sqrt{1-k^2}$.

$$= \sqrt{1-k^2} - k \text{ Arccos } k$$

4: { 1: limits; 1: integrand; 1: antiderivative; 1: answer

(b) $k = \frac{1}{2} \Rightarrow A = \frac{\sqrt{3}}{2} - \frac{1}{2}\text{ Arccos}\frac{1}{2} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

1: answer

(c) In general, $A = \sqrt{1-k^2} - k \text{ Arccos } k$.

Then
$$\frac{dA}{dt} = \frac{-k\frac{dk}{dt}}{\sqrt{1-k^2}} - (\text{Arccos } k)\frac{dk}{dt} - k \cdot \frac{-1}{\sqrt{1-k^2}}\frac{dk}{dt}$$

$$= \frac{dk}{dt}\left[\frac{-k}{\sqrt{1-k^2}} - \text{Arccos } k + \frac{k}{\sqrt{1-k^2}}\right]$$

$$= (-\text{Arccos } k)\frac{dk}{dt}.$$

With $k = \frac{1}{2}$ and $\frac{dk}{dt} = \frac{1}{\pi}$, we obtain $\frac{dA}{dt} = -\frac{\pi}{3} \cdot \frac{1}{\pi} = -\frac{1}{3}$.

4: { 2: $\frac{dA}{dt}$; 1: substitution; 1: answer

Exam II
Section I
Part A — No Calculators

1. C p. 25

$f(x) = 4x^3 - 3x - 1 \Rightarrow f(2) = 32 - 6 - 1 = 25 \Rightarrow$ Point (2,25)

$f'(x) = 12x^2 - 3 \Rightarrow f'(2) = 48 - 3 = 45 \Rightarrow$ Slope = 45

$y - 25 = 45(x-2) \Rightarrow y = 45x - 65$

2. A p. 25

$$\int_0^1 \sin(\pi x)\,dx = -\tfrac{1}{\pi}\cos(\pi x)\Big]_0^1 = -\tfrac{1}{\pi}(\cos\pi - \cos 0) = -\tfrac{1}{\pi}(-1-1) = \tfrac{2}{\pi}$$

3. B p. 26

This is the definition of the derivative of the cosine function at coordinate x. Hence

$$\lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \cos'(x) = -\sin x$$

4. C p. 26

The graph is concave up when $y'' > 0$.

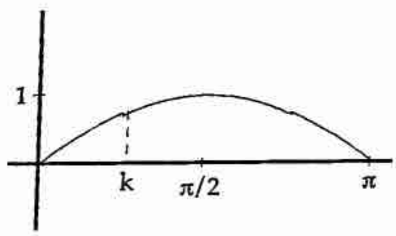
$y = x^5 - 5x^4 + 10x + 15$

$y' = 5x^4 - 20x^3 + 10$

$y'' = 20x^3 - 60x^2 = 20x^2(x-3)$

The factor $(20x^2)$ is always positive. The sign of y'' depends upon the sign of the factor $(x-3)$. $y'' > 0$ if and only if $x > 3$.

5. D p. 26



$$\int_0^\pi \sin x\,dx = -\cos x\Big]_0^\pi = -\cos\pi + \cos 0 = 2.$$

Thus the whole region has area 2.

We need to divide this into two parts, with the left-hand section having an area that is $\frac{1}{4}$ of the whole region. Hence we must find k so that $\int_0^k \sin x\,dx = \frac{1}{4}(2) = \frac{1}{2}$.

This gives: $-\cos x\Big]_0^k = \frac{1}{2} \Rightarrow -\cos k + 1 = \frac{1}{2} \Rightarrow \cos k = \frac{1}{2} \Rightarrow k = \frac{\pi}{3}$

6. C p. 27

$$\begin{aligned} x(t) &= (t-2)^3(t-6) \\ x'(t) &= 3(t-2)^2(t-6) + (t-2)^3 \\ &= (t-2)^2[\,3(t-6) + (t-2)\,] \\ &= (t-2)^2(4t-20) \\ &= 4(t-2)^2(t-5) \end{aligned}$$

This is positive-valued when $t > 5$.

---

7. E p. 27

$(x^2 - 1)^2 = x^4 - 2x^2 + 1$

$\int (x^4 - 2x^2 + 1)\,dx = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + C$

---

8. A p. 27

Note that the numerator is the derivative of the denominator. Hence

$$\int_{\pi/4}^{\pi/3} \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x|\Big]_{\pi/4}^{\pi/3} = \ln\left|\tan\frac{\pi}{3}\right| - \ln\left|\tan\frac{\pi}{4}\right|$$
$$= \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}.$$

This integration problem can also be done with a formal substitution.

Let $u = \tan x$. Then $du = \sec^2 x\,dx$.

In addition, since this is a definite integral, we can change the limits of integration.

When $x = \frac{\pi}{4}$, then $u = \tan\frac{\pi}{4} = 1$.

When $x = \frac{\pi}{3}$, then $u = \tan\frac{\pi}{3} = \sqrt{3}$.

Hence $\int_{\pi/4}^{\pi/3} \frac{\sec^2 x}{\tan x}\,dx = \int_1^{\sqrt{3}} \frac{du}{u} = \ln|u|\Big]_1^{\sqrt{3}} = \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}.$

---

9. B p. 28

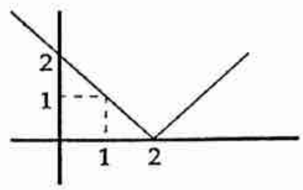
$$\lim_{x\to\infty} \frac{x^2-6}{2+x-3x^2} = \lim_{x\to\infty} \frac{1-\frac{6}{x^2}}{\frac{2}{x^2}+\frac{1}{x}-3} = -\frac{1}{3}.$$

---

10. D p. 28

$$\int_0^2 \sqrt{x^2 - 4x + 4} = \int_0^2 \sqrt{(x-2)^2} = \int_0^2 |x-2|\, dx$$

To evaluate this integral, count squares in the graph at the right, or note that the area of the triangle is $\frac{1}{2}(2)(2) = 2$.



11. C p. 28

$g(x) = \frac{x-2}{x+2}$

$g'(x) = \frac{(x+2)\cdot 1 - (x-2)\cdot 1}{(x+2)^2} = \frac{4}{(x+2)^2}$

Hence $g'(2) = \frac{4}{4^2} = \frac{1}{4}$.

12. B p. 29

$\frac{dy}{dx} = y^2$

$\frac{1}{y^2}\, dy = dx$

$-\frac{1}{y} = x + C$ At (-1, 1) $-1 = -1 + C$ and $C = 0$

$-\frac{1}{y} = x + 0 \Rightarrow y = -\frac{1}{x}$

The largest domain that contains (-1, 1) is $x < 0$.

Notice that this function does not have a derivative at $x = 0$, so the condition $x < 0$ must be added to the solution $y = -\frac{1}{x}$ in order for it to include $x = -1$ and be a solution of the original initial value problem.
In general, the solution of a differential equation must be a continuous function on an open interval containing the initial value x-value.

13. A p. 29

$f(x) = (2x-3)^4$

$f'(x) = 4(2x-3)^3 \cdot 2$

$f''(x) = 4 \cdot 3(2x-3)^2 \cdot 2^2$

$f'''(x) = 4 \cdot 3 \cdot 2(2x-3) \cdot 2^3$

$f^{(4)}(x) = 4! \cdot 2^4$

14. D p. 29

$$\int_2^4 (f(x)+3)\,dx = \int_2^4 f(x)\,dx + \int_2^4 3\,dx$$

$$= 6 + 3\cdot 2 = 12$$

---

15. A p. 30

$2xy + \sin y = 2\pi$ When $y=\pi$, then $2\pi x + 0 = 2\pi$, and $x = 1$

Using implicit differentiation we obtain $2y + 2xy' + \cos y\,\frac{dy}{dx} = 0.$

$\frac{dy}{dx} = \frac{-2y}{2x+\cos y}$ At $(1,\pi)$, $\frac{dy}{dx} = \frac{-2\pi}{2-1} = -2\pi.$

---

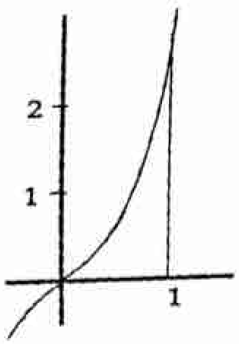
16. C p. 30

Solution I. $f(g(x)) = f(\ln x) = e^{2\ln x} = e^{\ln x^2} = x^2$
Then the derivative of $f(g(x))$ is $2x$.
The value of the derivative at $x = e$ is $2e$.

Solution II. $D_x[f(g(x))] = f'(g(x))\cdot g'(x)$.
At $x = e$, $g(x) = 1$.
Hence we want $f'(1)\cdot g'(e)$.
$f'(x) = 2e^{2x}$ and $g'(x) = \frac{1}{x}$, so $f'(1) = 2e^2$ and $g'(e) = \frac{1}{e}$.
Thus $D_x[f(g(x))]$, at $x = e$, has the value $2e^2\cdot\frac{1}{e} = 2e$.

---

17. C p. 30



$$\int_0^1 x e^{x^2}\,dx = \frac{1}{2}\int_0^1 (2x)e^{x^2}\,dx$$

$$= \frac{1}{2}e^{x^2}\Big]_0^1$$

$$= \frac{1}{2}(e-1)$$

---

18. E p. 31

$h'(x) = \frac{3}{4}(x^2-4)^{-1/4}\cdot(2x) = \frac{6}{4(x^2-4)^{1/4}}$

Then calculating $h'(2)$ involves a 0 in the denominator and a nonzero numerator, hence, $h'(2)$ does not exist.

19. B p. 31

$$y = \sqrt{x} - \frac{1}{x\sqrt[3]{x}} = x^{1/2} - x^{-4/3} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{4}{3}x^{-7/3}$$

---

20. D p. 31

In order for f to be continuous, we must have $\lim_{x\to 1} f(x) = f(1)$.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\sqrt{x+3}-\sqrt{3x+1}}{x-1}$$

$$= \lim_{x\to 1} \frac{\sqrt{x+3}-\sqrt{3x+1}}{x-1} \cdot \frac{\sqrt{x+3}+\sqrt{3x+1}}{\sqrt{x+3}+\sqrt{3x+1}}$$

$$= \lim_{x\to 1} \frac{(x+3)-(3x+1)}{(x-1)\sqrt{x+3}+\sqrt{3x+1}} = \lim_{x\to 1} \frac{-2x+2}{(x-1)\sqrt{x+3}+\sqrt{3x+1}}$$

$$= \lim_{x\to 1} \frac{-2(x-1)}{(x-1)\sqrt{x+3}+\sqrt{3x+1}} = \lim_{x\to 1} \frac{-2}{\sqrt{x+3}+\sqrt{3x+1}} = -\frac{1}{2}$$

Thus k must equal $-\frac{1}{2}$.

---

21. D p. 32

$$f(x) = \frac{x}{2x-3}$$

$$f'(x) = \frac{(2x-3) - x\cdot 2}{(2x-3)^2}$$

$$f'(1) = \frac{-1-2}{(-1)^2} = -3$$

The normal is a line that is perpendicular to the tangent at a point. Since the tangent line at $x = 1$ has a slope of -3, then the slope of the normal there must be $\frac{1}{3}$. Since $f(1) = -1$, the point at which the normal is to be drawn is $(1,-1)$.

Thus the equation of the line is:

$$y+1 = \frac{1}{3}(x-1)$$
$$3y+3 = x-1$$
$$4 = x-3y$$

22. A p. 32

With $f(x) = x\ln x$, we have $f'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$.
The critical number occurs when $f'(x) = 0$.

$1 + \ln x = 0 \Rightarrow \ln x = -1$
$\Rightarrow x = e^{-1}$

Evaluating the function f at $x = e^{-1}$, we have:
$f(e^{-1}) = e^{-1}\ln(e^{-1}) = e^{-1} \cdot (-1) = -\frac{1}{e}$.

To be sure that this is the minimum value, we check the concavity of the curve at $x = e^{-1}$.
$f''(x) = \frac{1}{x}$, so $f''(x)$ has a positive value at the positive number $x = e^{-1}$.
Hence the curve is concave up at the critical number $x = e^{-1}$, so there is a minimum value achieved there.

---

23. C p. 33

The slope field shows segments all with the same slope for a given y. That is, the slopes do no vary as x changes. Hence any suggested solution that has x in the formula for $\frac{dy}{dx}$ is incorrect. That allows the elimination of three of the proposed answers. If the correct answer were $\frac{dy}{dx} = y^2$, then the slopes of the segments would be at least 0 everywhere. They are not. Hence the correct answer is $\frac{dy}{dx} = -y$.

---

24. B p. 33

$$\frac{1}{\pi/4}\int_0^{\pi/4} \sec^2 x\,dx = \frac{4}{\pi}\tan x\Big]_0^{\pi/4} = \frac{4}{\pi}\cdot 1 = \frac{4}{\pi}$$

---

25. B p. 33

$g'(x) > 1$ assures us that g is strictly increasing. Then g will be one-to-one, and therefore will have an inverse.

---

26. E p. 34

Since f is continuous, positive at $x = 4$ and negative at $x = 5$, by the Intermediate Value Theorem, there is a point in the interval $[4, 5]$ where the function value is 0.

---

27. A p. 34

With $G(x) = \int_0^{2x} \cos(t^2)\,dt$, we first let $F(x) = \int_0^{x} \cos(t^2)\,dt$.

Then by the Second Fundamental Theorem, $F'(x) = \cos(x^2)$.

In addition, $G(x) = F(2x)$.

We use the Chain Rule to differentiate this.

$G'(x) = F'(2x) \cdot 2$.

Since $F'(x) = \cos(x^2)$, we have $F'(2x) = \cos((2x)^2) = \cos(4x^2)$.
Thus $G'(x) = 2\cos(4x^2)$ and $G'(\sqrt{\pi}) = 2\cos(4\pi) = 2$.

28. E p. 34

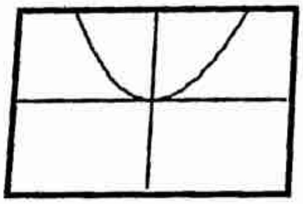
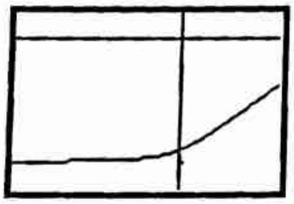
Velocity is the rate of change of the position with respect to time.
Hence we need to determine $\frac{dx}{dt}$.
Differentiate the given expression implicitly with respect to t.
$t \cdot \frac{dx}{dt} + x = 2x\frac{dx}{dt} \quad \Rightarrow \quad x = (2x - t)\frac{dx}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{x}{2x - t}$
When the particle is at $x = 2$, the time is determined by the equation
$tx = x^2 + 8$; in particular, $2t = 12$, so $t = 6$.
Then the velocity is evaluated by substituting $x = 2$ and $t = 6$ into the expression for $\frac{dx}{dt}$.
$\frac{dx}{dt} = \frac{2}{4-6} = -1$.

Exam II
Section I
Part B — Calculators Permitted

1. E P. 35

A function has a derivative at a particular x-coordinate if its graph is smooth there.

I. $y = \left|x^3 - 3x^2\right|$ II. $y = \sqrt{x^2 + .01} - \left|x - 1\right|$ III. $y = \frac{e^x}{\cos x}$


All three of these graphs are smooth at $x = 0$.

2. D p. 35

$$\text{Volume} = \int_0^5 20\,e^{.02t}\,dt = \frac{20}{.02}e^{.02t}\Big]_0^5 = 1000\,(e^{.1} - e^0) \approx 105 \text{ gal}$$

3. D p. 36

By the Quotient Rule, $f'(x) = \frac{(a + x^3)\cdot 6 - (6x)\cdot(3x^2)}{(a + x^3)^2} = \frac{6a - 12x^3}{(a + x^3)^2}$.

Then $f'(0) = \frac{6a}{a^2} = \frac{6}{a}$.

We are given that $f'(0) = 3$. Hence $\frac{6}{a} = 3$, so $a = 2$.

4. C p. 36

$$f(x) = \frac{(x-1)^2}{2x^2 - 5x + 3} = \frac{(x-1)^2}{(2x-3)(x-1)} = \frac{x-1}{2x-3} \quad \text{if } x \neq 1.$$

I. $f(1)$ does not exist; hence f is **not** continuous at $x = 1$. False

II. $\lim_{x\to 1} f(x) = 0$; thus there is **not** a vertical asymptote at $x = 1$.
There is a vertical asymptote at $x = \frac{3}{2}$. False

III. $\lim_{x\to\infty} f(x) = \frac{1}{2}$ True

5. C p. 36

Solution I. By the Chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Since $\frac{dy}{du} = 1 + 2e^u$ and $\frac{du}{dx} = \frac{1}{x}$, then $\frac{dy}{dx} = (1 + 2e^u) \cdot \frac{1}{x}$.

When $x = \frac{1}{e}$, we have $u = 1 + \ln\frac{1}{x} = 1 - 1 = 0$.

With those values: $\frac{dy}{dx} = (1 + 2e^0) \cdot e = 3e$

Solution II. Substitution gives $y = (1 + \ln x) + 2e^{1 + \ln x} = 1 + \ln x + 2ex$

Then $\frac{dy}{dx} = \frac{1}{x} + 2e$, so when $x = \frac{1}{e}$, $\frac{dy}{dx} = 3e$.

---

6. A p. 37

Since the radius is always 3 times the height, , we have $h = \frac{1}{3}r$, so

$V = \frac{1}{3}\pi r^2(\frac{1}{3}r) = \frac{1}{9}\pi r^3$. Then $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dr}{dt}$.

On the graph, we see that when $t = 6$, then $V = 4\pi$. In addition, we can approximate the slope of the curve, $\frac{dV}{dt}$, at the point where $t = 6$.

$\frac{dV}{dt} \approx \frac{1.6\pi}{2} = 0.8\pi$.

Knowing that the volume is 4π when $t = 6$ allows us to compute the radius of the conical pile at that moment. $4\pi = \frac{1}{9}\pi r^3 \Rightarrow r = 36^{1/3}$.

Substituting these values into the equation $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dr}{dt}$, we have:

$0.8\pi = \frac{1}{3}\pi\, 36^{2/3} \frac{dr}{dt}$, which gives $\frac{dr}{dt} = \frac{2.4\pi}{36^{2/3}\pi} \approx 0.22$

---

7. E p. 37

I. $f(1) = 1, \quad f'(3) \approx -2$ True

II. $\int_1^2 f(x)\,dx \approx 6.5, \quad f'(3.5) = 0$ True

III. $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = f'(2) \approx 1$

$\frac{f(2.5) - f(2)}{2.5 - 2} \approx \frac{2.5 - 2.3}{2.5 - 2} = \frac{.2}{.5} = .4$ True

8. A p. 37

Solution I. Starting with $5x^3 + 40 = \int_a^x f(t)\,dt$, differentiate to obtain $15x^2 = f(x)$.

Then $\int_a^x 15t^2\,dt = 5t^3\Big]_a^x = 5x^3 - 5a^3$

Since we are given that $5x^3 + 40 = \int_a^x f(t)\,dt$,

we can set $5x^3 + 40 = 5x^3 - 5a^3$.

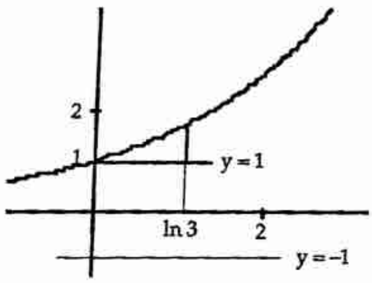
Then $-5a^3 = 40$, so $a = -2$.

Solution II. Let $x = a$, then $5a^3 + 40 = \int_a^a f(t)\,dt = 0$.

Then $5a^3 + 40 = 0$, and $a = -2$.

---

9. A p. 38



Use the washer method.

$$V = \pi\int_0^{\ln 3}[(e^{x/2}+1)^2 - 2^2]\,dx$$

$$\approx 5.128.$$

---

10. A p. 38

$$\frac{dy}{dx} = \frac{x\sin(x^2)}{y}$$

$$y\,dy = x\sin(x^2)\,dx$$

$$\int y\,dy = \int x\sin(x^2)\,dx$$

$$\frac{y^2}{2} = -\tfrac{1}{2}\cos(x^2) + C$$

$$y^2 = -\cos(x^2) + D$$

$x = 0,\ y = 1 \Rightarrow 1 = -\cos(0) + D \Rightarrow D = 2$

$$y = \sqrt{2 - \cos(x^2)}$$

---

11. D p. 38

$y = \sin u \qquad u = v - \frac{1}{v} \qquad v = \ln x$

$\frac{dy}{du} = \cos u \qquad \frac{du}{dv} = 1 + \frac{1}{v^2} \qquad \frac{dv}{dx} = \frac{1}{x}$

$x = e \Rightarrow v = 1 \Rightarrow u = 0$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

$$= (\cos u)\left(1 + \tfrac{1}{v^2}\right)\left(\tfrac{1}{x}\right) = 1\cdot 2\cdot\tfrac{1}{e} = \tfrac{2}{e}$$

12. C p. 39

The average values of the functions f and g on [0,b] are

$$\frac{1}{b}\int_0^b \cos(2x)\,dx \text{ and } \frac{1}{b}\int_0^b (e^x - 1)\,dx .$$

Setting these equal, we have

$$\frac{1}{b}\int_0^b \cos(2x)\,dx = \frac{1}{b}\int_0^b (e^x - 1)\,dx$$

$$\int_0^b \cos(2x)\,dx = \int_0^b (e^x - 1)\,dx$$

$$\frac{1}{2}\sin(2x)\Big|_0^b = (e^x - x\Big|_0^b$$

$$\frac{1}{2}\sin(2b) = (e^b - b) - (1 - 0)$$

$$\frac{1}{2}\sin(2b) = e^b - b - 1$$

The intersection of the graphs of $y_1 = \frac{1}{2}\sin(2x)$ and $y_2 = e^x - x - 1$ occurs at $x \approx 0.854$.

Alternatively, the zero of $y = \frac{1}{2}\sin(2x) - e^x - x - 1$ occurs at $x \approx 0.854$.

---

13. D p. 39

I. The function f has a horizontal tangent at each x-coordinate where $f'(x) = 0$. This is the case at $x = -3$, 1, and 3, but **not** at $x = 2$. False

II. Since $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$, the function f is decreasing to the left of $x = 1$ and increasing to the right. True

III. Since $f'(x)$ is increasing to the left of $x = -3$ and decreasing to the right of $x = -3$, the concavity if the graph of f will change at $x = -3$. True

---

14. D p. 40

$f(x) \approx f(a) + f'(a)(x - a)$

When $x = 2.8$, $a = 3$, $f(a) = 5$ and $f'(a) = 4$, then

$f(2.8) \approx 5 + 4(2.8 - 3) = 4.2.$

15. B p. 40

$N(t) = 200 \ln(t^2 + 36)$

$N'(t) = 200 \cdot \dfrac{1}{t^2 + 36} \cdot 2t$

The graph of $N'(t)$ has a maximum at $t = 6$ days.

Alternatively, $N''(t) = \dfrac{(t^2 + 36)(400) - 400t(2t)}{(t^2 + 36)^2} = \dfrac{14400 - 400t^2}{(t^2 + 36)^2}$.

$N''(t) = 0$ when $t^2 = 36$ and $t = 6$ days.

---

16. C p. 41

$a(t) = 4e^{2t} \quad \Rightarrow \quad v(t) = 2e^{2t} + C$

When $t = 0$, then $v = -2$.

In the last equation, this gives $-2 = 2e^0 + C$, so that $C = -4$.

Then $v(t) = 2e^{2t} - 4$.

Antidifferentiating again, we have $x(t) = e^{2t} - 4t + D$.

When $t = 0$, then $x = 2$.

This gives $2 = e^0 - 4 \cdot 0 + D$, so that $D = 1$.

Then $x(t) = e^{2t} - 4t + 1$, and we obtain the particular value $x(\frac{1}{2}) = e - 1$.

---

17. D p. 41

I.	On the interval $(-2,-1)$, $f'(x) > 0$. Hence f is increasing.	True
II.	At $x = 0$, $f'(x)$ changes from decreasing to increasing.	True
III.	On the interval $(-1, 0)$, $f'(x)$ is decreasing, so f is concave down.	False

Exam II
Section II
Part A — Calculators Permitted

1. p. 43

The curves intersect at A = 1.102.

(a) $\int_0^A [(4-x^2)-(1+2\sin x)]\,dx \approx 1.764$

(b) $\pi\int_0^A \left[(4-x^2)^2 - (1+2\sin x)^2\right]dx \approx 30.460$

(c) $\int_0^A \left[(4-x^2)-(1+2\sin x)\right]^2 dx \approx 3.671$

1: Correct limits in an integral in (a), (b) and (c)

$2:\begin{cases}1:\text{integrand}\\1:\text{answer}\end{cases}$

$3:\begin{cases}2:\text{integrand + constant}\\1:\text{answer}\end{cases}$

$3:\begin{cases}2:\text{integrand}\\1:\text{answer}\end{cases}$

2. p. 44

(a) $\int_0^{24} R(t)\,dt \approx 4\,[R(4)+R(8)+R(12)+R(16)+R(20)+R(24)]$
$= 4(28+33+42+46+50+52)$
$= 1004$ gallons

Approximation is greater than the true value be cause the function is increasing on the interval.

$3:\begin{cases}1:\text{Right - hand sum}\\1:\text{answer}\\1:\text{Explanation}\end{cases}$

(b)

Average value $= \frac{1}{24}\int_0^{24} 27e^{0.03t}\,dt \approx 39.541$

$3:\begin{cases}1:\text{average value constant}\\1:\text{limits and integrand}\\1:\text{answer}\end{cases}$

(c)

$w(24) = w(0) + \int_0^{24} 27e^{0.03t}\,dt$
$= 125 + 948.990$
$= 1073.99$ gallons

$3:\begin{cases}1:\text{initial condition}\\1:\text{limits and integrand}\\1:\text{answer}\end{cases}$

Exam II
Section II
Part B — No Calculators

3. p. 45

(a) $F(x) = \int_1^{\sqrt{x}} \frac{2t-1}{t+2}\,dt \Rightarrow F(1) = \int_1^1 \frac{2t-1}{t+2}\,dt = 0.$

1: answer

(b) $F'(x) = \frac{2\sqrt{x}-1}{\sqrt{x}+2}\cdot\frac{1}{2\sqrt{x}} \Rightarrow F'(1) = \frac{2-1}{1+2}\cdot\frac{1}{2} = \frac{1}{6}.$

$2: \begin{cases} 1: F'(x) \\ 1: F'(1) \end{cases}$

(c) slope $= F'(1)$ and $F(1) = 0$

$y - 0 = \frac{1}{6}(x-1)$

$3: \begin{cases} 1: \text{slope} \\ 1: \text{point} \\ 1: \text{tangent equation} \end{cases}$

(d) F increases when $F'(x) > 0$.

$F'(x) = \frac{2\sqrt{x}-1}{\sqrt{x}+2}\cdot\frac{1}{2\sqrt{x}} = 0$ when $2\sqrt{x} - 1 = 0$

$\sqrt{x} = \frac{1}{2}$

$x = \frac{1}{4}$

F increases when $x > \frac{1}{4}$.

$3: \begin{cases} 1: F'(x) > 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

4. p. 46

(a) $a(3.5) = v'(3.5) = \frac{2-(-2)}{2-6} = -1$ miles / min^2	2: { 1: answer; 1: units
(b) The car changes direction at $t = 4$ and $t = 7$ because the velocity changes sign at each of these points in time.	2: { 1: answer; 1: justification
(c) The total distance traveled by the car is the sum of the areas of the regions between the velocity graph and the x-axis. Total distance $= \int_0^8 \lvert v(t)\rvert\,dt = 4 + \lvert -3\rvert + 1 = 8$	2: { 2: $\int_0^8 \lvert v(t)\rvert\,dt$; 1: answer
(d) The average rate of change of v over the interval [3, 8] is $\frac{v(8)-v(3)}{8-3} = \frac{2-1}{5} = 1/5$ miles / min^2. No, the Mean Value Theorem does not apply to v on the interval [3, 8] because v is not differentiable at $t = 6$.	3: { 1: difference quotient; 1: answer; 1: anwer with explanation

5. p. 47

(a) Point = (0 ,5); slope = $f'(0) = 3$; tangent line : $y - 5 = 3x$	2: { 1: point / slope; 1: tangent line equation
(b) f has a relative maximum at $x = 2$. There is only one x-value where f' changes from positive to negative.	2: { 1: answer; 1: justification
(c) f' changes from increasing to decreasing or vice versa at $x = 0$ and $x = 4$. Thus, the graph of f has points of inflection at $x = 0$ and $x = 4$.	2: { 1: answer; 1: justification
(d) Candidates for absolute maximum are where f' changes from positive to negative ($x = 2$) and the endpoints ($x = -6, 6$). $f(2) = f(0) + \int_0^2 f'(x)\,dx = 5 + 3 = 8$; $f(6) = f(0) + \int_0^6 f'(x)\,dx = 5 + (3 - 2\pi) = 8 - 2\pi \approx 2$; $f(-6) = f(0) - \int_{-6}^0 f'(x)\,dx = 5 - (-2 + 3) = 4$. The absolute maximum of $f(x)$ on the interval [-6, 6] is $f(2) = 8$.	3: { 1: identifies $x = 2$ as candidate; 1: endpoints; 1: value and explanatio

6. p. 48

(a) $f(x) = ax + \frac{b}{x}$, where a and b are positive.

$f'(x) = a - \frac{b}{x^2}$

$f'(x) > 0 \Rightarrow ax^2 - b > 0$

$\Rightarrow x^2 > \frac{b}{a}$

$\Rightarrow x < -\sqrt{\frac{b}{a}}$ or $x > \sqrt{\frac{b}{a}}$

3: { 1: $f'(x) > 0$; 1: Critical points; 1: answer }

(b) There is a relative maximum at $x = -\sqrt{\frac{b}{a}}$; $y = a\left[\sqrt{\frac{b}{a}}\right] + \frac{b}{\sqrt{\frac{b}{a}}}$

$= -2\sqrt{ab}$.

There is a relative minimum at $x = \sqrt{\frac{b}{a}}$; $y = a\left[\sqrt{\frac{b}{a}}\right] + \frac{b}{\sqrt{\frac{b}{a}}}$

$= 2\sqrt{ab}$.

2: { 1: Coordinates of max; 1: Coordinates of min }

(c) $f''(x) = \frac{2b}{x^3} > 0$ when $x > 0$.

2: { 1: $f''(x) > 0$; 1: answer }

(d) $f''(x)$ never changes sign at a point on the curve. Although the curve is concave down for $x < 0$ and the curve is concave up for $x > 0$, there is not a point of inflection at $x = 0$ since there is no point on the curve there.

2: { 1: answer; 1: justification }

Exam III
Section I
Part A — No Calculators

1. A p. 49

$g'(x) = \cos(\sin x)$
$g''(x) = -\sin(\sin x) \cdot \cos x$

$g'(0) = \cos(\sin 0) = \cos 0 = 1$. Since $g'(0) > 0$, g is increasing at $x = 0$.
$g''(0) = -\sin(\sin 0) \cdot \cos 0 = 0$. Then g is not concave down at $x = 0$, because g' is not decreasing at $x = 0$.
g is increasing at $x = 0$, so g cannot have a relative maximum there.

The only true statement is (I).

2. C p. 49

$$\int_0^5 \frac{dx}{\sqrt{1+3x}} = \frac{1}{3} \cdot \int_0^5 \frac{3dx}{\sqrt{1+3x}} = \frac{2}{3}\sqrt{1+3x}\,\Big]_0^5 = \frac{2}{3}(4-1) = 2$$

3. D p. 50

An equation of the tangent line that passes through the point (4, 1) with slope $f'(4) = 5$ is $y - 1 = 5(x-4)$. When $y = 0$, $0 - 1 = 5(x-4) \Rightarrow -1 = 5x - 20 \Rightarrow 5x = 19 \Rightarrow x = 3.8$

4. E p. 50

$$\int_0^2 e^{-x}\,dx = -e^{-x}\Big]_0^2 = -\frac{1}{e^2} + 1 = 1 - \frac{1}{e^2}$$

5. C p. 50

$g(x) = x + \cos x$
By definition, $\lim\limits_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$
Hence the value of the limit is $g'(x) = 1 - \sin x$.

6. C p. 51

$$\int_0^4 \frac{2x}{x^2+9}\,dx = \ln\left|x^2+9\right|\Big]_0^4 = \ln 25 - \ln 9 = \ln\left[\frac{25}{9}\right]$$

7. D p. 51

By definition, $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$

$= \lim_{h\to 0} \frac{4xh+2h^2}{h}$ (Since $g(x+h) - g(x) = 4xh + 2h^2$)

$= \lim_{h\to 0} (4x+2h) = 4x$

---

8. D p. 51

I. $f(4) = f(0) + \int_0^4 f'(x)\,dx = 3 + 0 = 3$ — True

II. On the interval (0, 5), the graph of f has both a positive and negative slope. — False

III. f' changes from increasing to decreasing or vice versa at $x = 0$ and $x = 5$. Thus, the graph of f has points of inflection points at $x = 0$ and $x = 5$. — True

---

9. C p. 52

The average value of f on $[a, b]$ is defined to be $\frac{1}{b-a}\int_a^b f(x)\,dx$. Therefore

average value $= \frac{1}{3-(-3)}\int_{-3}^{3}(1 + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + 1 + 2)\,dx = \frac{1}{6}\left(\frac{7}{6}\right) = \frac{7}{12}$. The answer is C.

---

10. E p. 52

$v(t) = \frac{1}{1+t}$ $\quad s(t) = \int v(t)\,dt = \ln|1+t| + C$

$s(0) = 5 \Rightarrow C = 5$

$s(t) = \ln|1+t| + 5 \Rightarrow s(3) = \ln(4) + 5$

---

11. A p. 52

Solution I. To find the inverse of the function $y = g(x) = \sqrt[3]{x-1}$, interchange x and y and solve for y.

$x = \sqrt[3]{y-1} \Rightarrow x^3 = y - 1 \Rightarrow y = x^3 + 1.$

Thus $f(x) = g^{-1}(x) = x^3 + 1.$

Then $f'(x) = 3x^2.$

Solution II. Since f is the inverse of g, we have $f(g(x)) = x$.

Differentiating gives: $f'(g(x)) \cdot g'(x) = 1$

Then $f'(g(x)) = \frac{1}{g'(x)}$.

Since $g(x) = (x-1)^{1/3}$, we know $g'(x) = \frac{1}{3}(x-1)^{-2/3}$.

Hence $f'(g(x)) = 3(x-1)^{2/3}$.

Since $g(x) = (x-1)^{1/3}$, this is: $f'((x-1)^{1/3}) = 3(x-1)^{2/3}$.

If we now substitute u for $(x-1)^{1/3}$, this is $f'(u) = 3u^2$.

12. A p. 53

Define the function G by $G(x) = \int_0^x \sqrt[3]{1+t^3}\,dt$.

Then by the Second Fundamental Theorem, $G'(x) = \sqrt{1+x^3}$
Note that $F(x) = G(\cos x)$, so we use the Chain Rule to determine $F'(x)$.
$F'(x) = G'(\cos x)\cdot[-\sin x]$
Then $F'(\frac{\pi}{2}) = G'(\cos\frac{\pi}{2})\cdot(-\sin\frac{\pi}{2}) = G'(0)\cdot(-1) = -1$.

---

13. C p. 53

The slope of $y = 3x+2$ is $m = 3$. Find the first quadrant point on the curve $y = x^3+k$ at which the slope is 3.

$y' = 3x^2 \quad\Rightarrow\quad 3x^2 = 3 \quad\Rightarrow\quad x^2 = 1 \quad\Rightarrow\quad x = \pm 1$

Since we need a first quadrant point, $x = 1$, and the point on the line is $P(1,5)$. Then $y = x^3+k$ must pass through $(1,5)$, so $k = 4$.

---

14. D p. 53

I.	A solution containing (0,2) never gets below the y-value of 1.	False
II.	From both above and below, as $y \to 1$, $\frac{dy}{dx} \to 0$.	True
III.	At a given value of y, $\frac{dy}{dx}$ is constant.	True

---

15. B p. 54

$$\frac{d}{dx}[\text{Arctan}(3x)] = \frac{1}{1+(3x)^2}\cdot 3 = \frac{3}{1+9x^2}$$

---

16. E p. 54

$$\lim_{x\to 1}\frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1}\frac{(x+3)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1}\frac{x+3}{x+1} = \frac{4}{2} = 2$$

---

17. C p. 54

g is an antiderivative of f. By the Fundamental Theorem,

$$\int_a^b g'(x)\,dx = g(b) - g(a). \text{ Thus, } \int_2^3 f(x)\,dx = \int_2^3 g'(x) = g(3) - g(2).$$

---

18. C p. 55

$y = 2e^{\cos x}$

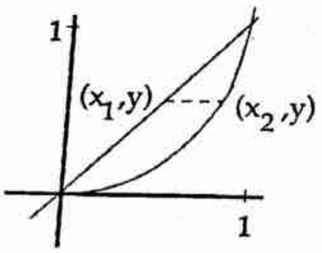
$\frac{dy}{dt} = 2e^{\cos x}(-\sin x)\frac{dx}{dt}$

When $x = \frac{\pi}{2}$ and $\frac{dy}{dt} = 5$, then $5 = 2e^0(-1)\frac{dx}{dt}$. Hence $\frac{dx}{dt} = -\frac{5}{2}$.

19. A p. 55

$$\int_1^2 \frac{dx}{x^3} = -\frac{1}{2x^2}\Big]_1^2 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$$

20. D p. 55



The horizontal distance is the difference between the x-coordinates at a particular y.

Thus $D = x_2 - x_1 = \sqrt{y} - y$.

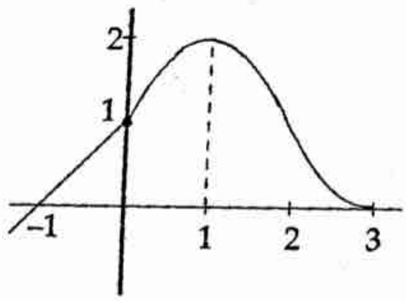
To maximize this distance function, differentiate and set equal to 0.

$$\frac{dD}{dy} = \frac{1}{2\sqrt{y}} - 1 = 0 \quad \Rightarrow \quad 2\sqrt{y} = 1$$

Thus the critical number is $y = \frac{1}{4}$.

The distance, for $y = \frac{1}{4}$, is $D = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

21. D p. 56



$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx.$$

$$\int_{-1}^{0} f(x)\,dx = \frac{1}{2}$$ (The area of the triangle)

$$\int_0^1 f(x)\,dx = \int_0^1 (1 + \sin \pi x)\,dx = x - \frac{1}{\pi}\cos \pi x\Big]_0^1$$

$$= \left[1 + \frac{1}{\pi}\right] - \left[0 - \frac{1}{\pi}\right] = 1 + \frac{2}{\pi}$$

The total of these two integrals is $\frac{3}{2} + \frac{2}{\pi}$.

22. C p. 56

We use implicit differentiation to obtain $g'(x)$ from

$g(x) = x\sqrt{f(x)}$.

$g'(x) = \sqrt{f(x)} + x \cdot \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$

At $x = 3$, $f(3) = 4$, $f'(3) = 8$ and $g(3) = 3\sqrt{4} = 6$.

Thus, $g'(3) = \sqrt{4} + 3 \cdot \frac{1}{2\sqrt{4}}\ 8 = 2 + 6 = 8$, and the equation of the tangent at ((3, 6) is

$y - 6 = 8(x - 3)$.

---

23. E p. 56

$$f(x) = x^{2/3}(5 - 2x)$$
$$f'(x) = \tfrac{2}{3}x^{-1/3}(5 - 2x) - 2x^{2/3}$$
$$= \tfrac{2}{3}x^{-1/3}[(5 - 2x) - 3x]$$
$$= \tfrac{2}{3}x^{-1/3}[5 - 5x]$$
$$= \tfrac{10}{3}x^{-1/3}(1 - x)$$

This is positive when $0 < x < 1$.

---

24. D p. 57

Differentiate the expression $\ln(x + y) = x^2$ implicitly. (Watch out for the chain rule!)

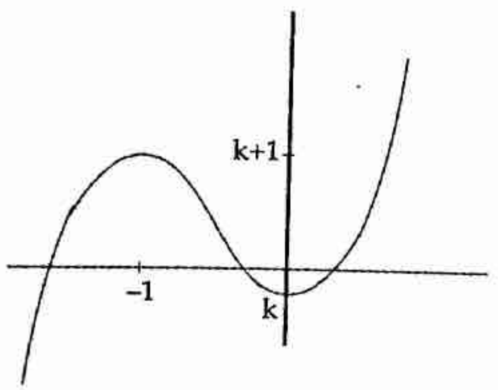
$$\frac{1}{x + y} \cdot (1 + y') = 2x$$

$(1 + y') = 2x^2 + 2xy \quad \Rightarrow \quad y' = 2x^2 + 2xy - 1$

At $x = 1$, $\ln(1 + y) = 1 \quad \Rightarrow \quad (1 + y) = e$ so that $y = e - 1$.

Then $y' = 2(1) + 2(1)(e - 1) - 1 = 2e - 1$

25. D p. 57



$y = 2x^3 + 3x^2 + k$

The constant k only affects the vertical location of the graph of the cubic. We must adjust k so that the relative maximum and minimum points are on opposite sides of the x-axis.

$\frac{dy}{dx} = 6x^2 + 6x = 0$

The critical numbers are 0 and −1.
$y(0) = k$ and $y(-1) = 1 + k$.
We must have $k < 0$ and $k + 1 > 0$. Thus $-1 < k < 0$.

---

26. C p. 57

Since the inputs are not equally spaced, we must calculate the area of the individual trapezoids.

$$\text{Distance} = \frac{1}{2}(3)[10 + 14] + \frac{1}{2}(2)[14 + 20] + \frac{1}{2}(4)[20 + 22] = 154 \text{ ft}$$

---

27. C p. 58

The particle is moving to the right if the first derivative is positive.
$x'(t) = 3\cos^2 t \cdot [-\sin t]$
Then $x'(t) > 0$ if $\sin t < 0$. This first happens if $\pi < t < \frac{3\pi}{2}$.

---

28. E p. 58

$\frac{dy}{dx} = \frac{x}{y}$ Separating the variables gives

$y\,dy = x\,dx$ Integrating yields

$\frac{y^2}{2} = \frac{x^2}{2} + C$ At (-2, -1), $\frac{1}{2} = \frac{4}{2} + C$ and $C = -\frac{3}{2}$.

$\frac{y^2}{2} = \frac{x^2}{2} - \frac{3}{2} \Rightarrow y^2 = x^2 - 3 \Rightarrow y = \pm\sqrt{x^2 - 3}$ where $x > \sqrt{3}$ or $x < -\sqrt{3}$.

Since (-2, -1) satisfies the solution, then $y = -\sqrt{x^2 - 3}$ and $x < -\sqrt{3}$.

Exam III
Section I
Part B — Calculators Permitted

1. A p. 59

Using your calculator find the intersection of the graphs of the derivatives: $y_1 = 2x + \frac{1}{e^x}$ and $y_2 = \frac{1}{\sqrt{x}}$; the curves intersect at $x = 0.435$.

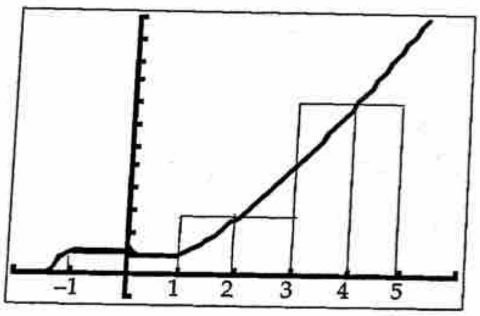
2. B p. 59

The average rate of change of a function f on the interval $[1,3]$ is $\frac{f(3)-f(1)}{3-1}$. For this function,

$$\frac{f(3)-f(1)}{3-1} = \frac{\int_0^3 f(t)\,dt - \int_0^1 f(t)\,dt}{2}$$

$$= \frac{1}{2}\int_1^3 f(t)\,dt \approx 0.232$$

3. B p. 60



The width of each of the three rectangles is 2. Since we are forming a sum using midpoints, we evaluate the function at x = 0, 2, and 4.

x	0	2	4
f(x)	1	2.6458	7.8102

The midpoint approximation is:
$M_3 = 2[1 + 2.6458 + 7.8102] = 22.912$

4. D p. 60

The average rate of change of f on the interval [3,x] is defined to be $\frac{f(x)-f(3)}{x-3}$.

In this problem, we have $\frac{f(x)-f(3)}{x-3} = \frac{x^2-x-6}{x-3}$.

Also by an alternate definition of the derivative, $f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$.

Hence $f'(3) = \lim_{x\to 3} \frac{x^2-x-6}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3}(x+2) = 5$.

5. E p. 60

Consider the function $y = \frac{\sin x}{x}$.

I.	It has a removable discontinuity at $x = 0$.	False
II.	$\lim_{x\to\infty} \frac{\sin x}{x} = 0$.	True
III.	It has zeros at $x = \pm n\pi$, where n is an integer.	True

6. C p. 61

The graph of $y = f(x+1)$	is the graph of f shifted one unit left.	IV
The graph of $y = f(x)+1$	is the graph of f shifted 1 unit up.	II
The graph of $y = f(-x)$	is the graph of f reflected in the y-axis.	III
The graph of $y = f'(x)$	is parabolic.	V

The only solution that starts IV, II, III, V is answer (C).

7. C p. 61

Volumes of revolution about the x-axis are easily done by the disk (washer) method: $V = \pi\int_a^b [f(x)]^2\,dx$. In this case, $f(x) = \sqrt{x}$.

$$V_{[0,4]} = \pi\int_0^4 \left(\sqrt{x}\right)^2 dx = \pi\int_0^4 x\,dx = \pi\cdot\left[\frac{x^2}{2}\right]_0^4 = 8\pi$$

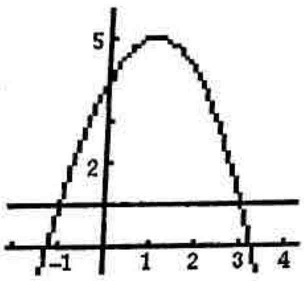
$$V_{[0,k]} = \pi\int_0^k \left(\sqrt{x}\right)^2 dx = \pi\int_0^k x\,dx = \pi\cdot\left[\frac{x^2}{2}\right]_0^k = \frac{\pi k^2}{2}$$

We need $\frac{\pi k^2}{2} = \frac{1}{2}(8\pi)$. Thus $\frac{\pi k^2}{2} = 4\pi$, so $k^2 = 8$, and $k = 2\sqrt{2} \approx 2.828$.

8. D p. 62

I.	f is decreasing for $-2 < x < -1$ since $f'(x) < 0$ there.	False
II.	$f'(0)$ exists, so f is continuous at $x = 0$.	True
III.	$f'(x)$ has a minimum at $x = -2$, with $f''(-2) = 0$.	True

9. C p. 62



First determine the intersection points of the two functions.

$-x^2 + 2x + 4 = 1$

$0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$x = -1, 3$

The area is then

$\int_{-1}^{3}$ (top function – bottom function) dx.

$$\int_{-1}^{3} ((-x^2 + 2x + 4) - 1)\,dx = \int_{-1}^{3} (-x^2 + 2x + 3)\,dx \approx 10.667$$

---

10. C p. 62

$f'(x) = g'(x) \Rightarrow f(x) - g(x) = C$

$f(1) = 2$ and $g(1) = 3 \Rightarrow f(x) - g(x) = -1$

The graphs do not intersect, since the graph of f is always 1 unit below the graph of g.

---

11. C p. 63

I. Ave. rate of change $= \frac{f(3) - f(-2)}{3 - (-2)} = \frac{2 - (-1)}{3 + 2} = \frac{3}{5}$ False

II. At the point (2,3), the tangent line is horizontal. True

III. The 4-subinterval left-sum approximation to $\int_{-1}^{3} f(x)\,dx$ has common width 1 and function values $-1, 0, 2, 3$; the approximation is $1 \cdot [-1 + 0 + 2 + 3] = 4$. True

---

12. B p. 63

The distance between the ships at time t is given by

$D(t) = \sqrt{W^2(t) + S^2(t)}$ This can be written $D^2 = W^2 + S^2$, where it is understood that all variables are functions of time t. Differentiating implicitly with respect to t we obtain

$$2D \cdot D' = 2W \cdot W' + 2S \cdot S', \quad \text{then} \quad D' = \frac{W \cdot W' + S \cdot S'}{D}.$$

When $t = 1$, we read from the graphs that $W = 5$ and $S = 4$.
We can also approximate the slopes of the two curves at the points where $t = 1$.

$W'(1) = \frac{1}{1/2} = 2$ and $S'(1) = \frac{2}{1/2} = 4$. (Note the different scale for t.)

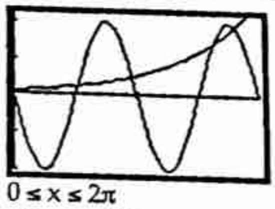
Thus $D'(1) = \frac{(5)(2) + (4)(4)}{\sqrt{5^2 + 4^2}} = 4.06$ knots.

13. E p. 64

$x_1'(t) = -2\sin(2t)$

$x_2'(t) = \frac{1}{2}e^{(t-3)/2}$

Graph these two velocity functions.
There are four intersection points.



$0 \le x \le 2\pi$
$-2 \le y \le 2.2$

---

14. C p. 64

The line $x - 2y + 9 = 0$ has slope $m = \frac{1}{2}$. Since it is parallel to the line through $(1, f(1))$ and $(5, f(5))$, we know that $\frac{f(5) - f(1)}{5 - 1} = \frac{1}{2}$.

Since $f(1) = 2$, we then have: $\frac{f(5) - 2}{4} = \frac{1}{2}$.

Thus $f(5) - 2 = 2$, so $f(5) = 4$.

[Note: The differentiability of f, the point $(3,6)$, and the tangency of the line to the graph of f are all irrelevant.]

---

15. A p. 65

Remember to use the Chain Rule.

$$\frac{d}{dx}f(x^2) = f'(x^2)\cdot 2x = 2x\cdot g(x^2)$$

$$\frac{d^2}{dx^2}f(x^2) = 2g(x^2) + 2x\cdot g'(x^2)\cdot 2x$$

$$= 2g(x^2) + 4x^2 f(3x^2)$$

---

16. B p. 65

Separate variables in the differential equation.

$$\frac{dy}{dx} = 4x\sqrt{y} \Rightarrow \frac{dy}{\sqrt{y}} = 4x\,dx$$

$$\Rightarrow 2\sqrt{y} = 2x^2 + C$$

Since the point $(1,9)$ is on the graph, we obtain

$$2\sqrt{9} = 2\cdot 1^2 + C$$
$$6 = 2 + C$$
$$C = 4$$

Thus $2\sqrt{y} = 2x^2 + 4$, or $\sqrt{y} = x^2 + 2$.

Then when $x = 0$, we have $\sqrt{y} = 2$, so $y = 4$.

17. E p. 65

Since g is continuoous and differentiable, there is at least one zero between x = 4 and x = 7. However, there could be others elsewhere.

g decreases from x = -2 to x = 1, then increases to x = 4. Thus there is at least one relative minimum between x = -2 and x = 4. However, there could be more.

g increases from x = 1 to x = 4, then decreases to x = 7. Thus there is at least one relative maximum between x = 1 and x = 7. However,there could be more.

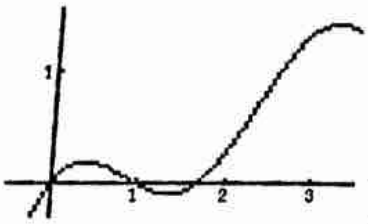
Since the concavity changes from concave up at the relative minimum to concave down at the relative maximum, there is at least one inflection point. However, there could be more.

(A) False There might be just one inflection point.
(B) False There might just be one zero.
(C) False There could be other zeros.
(D) False There might be only one relative maximum.
(E) True

Exam III
Section II
Part A — Calculators Permitted

1. p. 67

(a)



$f(x) = \ln(x+1) - \sin^2 x$ for $0 \le x \le 3.$

$y = 0 \quad \Rightarrow \quad \ln(x+1) - \sin^2 x = 0$

The function is pictured to the left. It has three zeros on $[0, 3]$.

With the capabilities of the graphing calculator, the zeros are found to be at

$x = 0$
$x = 0.964$
$x = 1.684$

3: three x - intercepts

(b) Consider the graph of the derivative of f: $f'(x) = \frac{1}{x+1} - 2\sin x \cos x.$ This derivative function is nonnegative valued on the intervals (0. 0.398) and (1.351, 3). Hence f is increasing there.

$3:\begin{cases} 1: f'(x) \ge 0 \\ 2: \text{answer} \end{cases}$

(c) The absolute maximum and absolute minimum values occur at either a critical point or an endpoint. The first derivative $f'(x) = \frac{1}{x+1} - 2\sin x \cos x$ has zeros at $x = 0.398$ and $x = 1.351$. At these critical points we have $f(0.398) = 0.184$ and $f(1.351) = -0.098$. At the endpoints we have $f(0) = 0$ and $f(3) = 1.366$. Therefore, on the interval [0, 3], the function's absolute minimum value is –0.098 and its absolute maximum value is 1.366.

$3:\begin{cases} 1: \text{identifies critical numbers and endpts as candidates} \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

2. p. 68

(a) $\frac{1}{2} \cdot 2 \cdot \left[(22 + 2 \cdot (14) + 2 \cdot (8) + 2 \cdot (5) + 2 \cdot (2) + 1\right] = 81 \text{ feet}$

$2:\begin{cases} 1: \text{trapezoid sum} \\ 1: \text{answer} \end{cases}$

(b) $\text{average} = \frac{1}{10}\int_0^{10} \left(0.18t^2 - 4t + 22\right)\, dt = 8$

$3:\begin{cases} 2: \text{integral} \\ \quad 1: \text{limits and } 1/(10-0) \\ \quad 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(c) $\frac{dv}{dt} = kv$

$\int \frac{1}{v}\frac{dv}{dt} = \int k\, dt$

$\ln v = kt + C$

$v = e^C \cdot e^{kt}$

$v(0) = 22 \Rightarrow 22 = e^C e^0 \Rightarrow e^C = 22$

To evaluate the constant k, use $f(2) = 14$.

$14 = 22e^{k\cdot 2} \Rightarrow k = \frac{1}{2}\ln\frac{14}{22}$

$v(t) = 22e^{\left(\frac{1}{2}\ln\frac{14}{22}\right)\cdot t} \approx 22e^{-0.226\cdot t}$

$4:\begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivatives} \\ 1: \text{uses initial conditions} \\ 1: \text{solves for } v \end{cases}$

Exam III
Section II
Part B — No Calculators

3. p. 69

(a) Area $= \int_{-1}^{3} (x+1)^{1/2}\,dx = \frac{2}{3}(x+1)^{3/2}\Big]_{-1}^{3}$

$= \frac{2}{3}[8] = \frac{16}{3}$

2: { 1: integral; 1: answer }

(b) Volume $= \pi\int_{-1}^{3} \left(\sqrt{x+1}+1\right)^2 - 1^2\,dx$

$= \pi\int_{-1}^{3} \left(x+1+2\sqrt{x+1}\right)dx$

$= \pi\left[\frac{x2}{2}+x+\frac{4}{3}(x+1)^{3/2}\right]_{-1}^{3}$

$= \pi\left[\left(\frac{9}{2}+3+\frac{4}{3}(4)^{3/2}\right)-\left(\frac{1}{2}-1\right)\right]$

$= \frac{56}{3}\pi$

4: { 1: limits and constant; 1: integrand; 1: antiderivative; 1: answer }

(c) Volume $= \frac{\pi}{8}\int_{-1}^{3} (x+1)\,dx$

3: { 2: integrand; 1: limits }

4. p. 70

(a) $f(x) = x^3 + px^2 + qx \Rightarrow f'(x) = 3x^2 + 2px + q \Rightarrow f''(x) = 6x + 2p$

$$\left\{\begin{array}{l} f(-1) = -8 \\ f'(-1) = 12 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} -8 = -1 + p - q \\ 12 = 3 - 2p + q \end{array}\right.$$

When we add these two equations, we obtain $4 = 2 - p$.
Thus $p = -2$. Substituting this value into one of the equations, we find that $q = 5$.

3: $\begin{cases} 1: f'(x) \\ 1: f(-1) \text{ and } f'(-1) \\ 1: \text{answer} \end{cases}$

(b) If the graph of f is to have a change in concavity at $x = 2$, then $f''(2) = 0$ and $f''(x)$ changes its sign at $x = 2$.

$f''(2) = 12 + 2p = 0 \Rightarrow p = -6$

Then $f''(x) = 6x - 12 = 6(x - 2)$. This <u>does</u> have a sign change at $x = 2$.

3: $\begin{cases} 1: f''(x) \\ 1: f''(2) \\ 1: \text{answer} \end{cases}$

(c) For f to be increasing everywhere, we must have $f'(x) > 0$ for all x.
Then $3x^2 + 2px + q > 0$ for all x.
$f'(x)$ is a quadratic function, opening upward.
$f'(x)$ will be positive valued $\Leftrightarrow$ f has no zeros
$\Leftrightarrow$ the discriminant of f is less than 0
$\Leftrightarrow$ $4p^2 - 12q < 0$
$\Leftrightarrow$ $p^2 < 3q$.

3: $\begin{cases} 1: f'(x) > 0 \\ 2: \text{answer} \end{cases}$

5. p. 71

(a) $y^3 - 3xy = 2$

Differentiating, we obtain $3y^2\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0.$

$$\frac{dy}{dx}(3y^2 - 3x) = 3y$$

$$\frac{dy}{dx} = \frac{3y}{3y^2 - 3x} = \frac{y}{y^2 - x}$$

2: $\begin{cases} 1\text{: implicit diff} \\ 1\text{: solves for } \frac{dy}{dx} \end{cases}$

(b) At the point (1,2), $\frac{dy}{dx}$ has the value $\frac{2}{4-1} = \frac{2}{3}$.

Therefore the tangent line has the equation $y - 2 = \frac{2}{3}(x - 1)$.

$y(1.3) = 2 + \frac{2}{3}(1.3 - 1) = 2.2$

2: $\begin{cases} 1\text{: tangent equation} \\ 1\text{: Use equation to approximate y at } x = 1.3 \end{cases}$

(c) $$\frac{d^2y}{dx^2} = \frac{(y^2 - x)\cdot\frac{dy}{dx} - y\left(2y\frac{dy}{dx} - 1\right)}{(y^2 - x)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,2)} = \frac{(4-1)\cdot\frac{2}{3} - 8\cdot\frac{2}{3} + 2}{9} = \frac{4 - \frac{16}{3}}{9} = -\frac{4}{27}$$

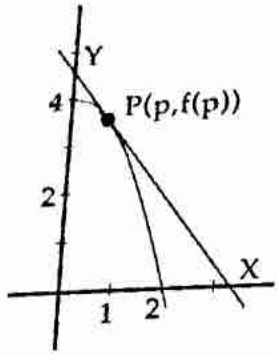
3: $\begin{cases} 2\text{: } \frac{d^2y}{dx^2} \\ 1\text{: answer} \end{cases}$

(d) Since $\frac{d^2y}{dx^2} < 0$ at the point (1, 2), the graph is concave down and the tangent line lies above the curve. Hence, the point (1.3, 2.2) is an overestimate.

2: $\begin{cases} 1\text{: answer} \\ 1\text{: justification} \end{cases}$

6. p. 72

(a)



The y-intercept (point Y) is $(0, p^2+4)$.

The x-intercept (point X) is $\left(\frac{p^2+4}{2p}, 0\right)$.

The area of the triangle is $A(p) = \frac{1}{2}\left(p^2+4\right)\left[\frac{p^2+4}{2p}\right]$

Then $A(2) = \frac{1}{2}(8)(2) = 8$.

4: { 1: x – intercept; 1: y – intercept; 1: expression for area of triangle; 1: answer }

(b) $A(p) = \frac{(p^2+4)^2}{4p}$.

Then $A'(p) = \frac{4p \cdot 2(p^2+4)\cdot 2p - (p^2+4)^2 \cdot 4}{16p^2} = \frac{4p^4+16p^2-p^4-8p^2-16}{4p^2}$

$= \frac{3p^4+8p^2-16}{4p^2}$

$= \frac{(3p^2-4)(p^2+4)}{4p^2}$

The critical numbers of the function A are at $p = \pm\frac{2}{\sqrt{3}}$.

The only critical number in the interval (0,2) is $p = \frac{2}{\sqrt{3}}$.

$\left(\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \approx \frac{2(1.7)}{3} = 1.15\right)$

$A'(1) < 0$ and $A'(2) > 0$, therefore by the First Derivative test, $A\left(\frac{2}{\sqrt{3}}\right)$ is a local minimum, and since there are no other critical points in the interval (0,2), $A\left(\frac{2}{\sqrt{3}}\right)$ is the absolute minimum.

5: { 2: $A'(p)$; 2: candidates for minimum; 1: answer }

Exam IV
Section I
Part A — No Calculators

1. A p. 73

$$\lim_{x\to 0} \frac{\frac{1}{x-1}+1}{x} = \lim_{x\to 0} \frac{1+(x-1)}{x(x-1)} = \lim_{x\to 0} \frac{x}{x(x-1)} = \lim_{x\to 0} \frac{1}{x-1} = -1$$

2. E p. 73

Since $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, the integrand has the form $\int e^u \, du$.

Thus $\int \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx = \int e^{\sqrt{x}} \frac{1}{2\sqrt{x}} dx = e^{\sqrt{x}} + C$

3. D p. 74

$y = \frac{3}{4+x^2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-3 \cdot 2x}{(4+x^2)^2} = \frac{-6x}{(4+x^2)^2}$

4. B p. 74

By the Second Fundamental Theorem, $\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x)$.

Then $F(x) = \int_1^x (\cos 6t + 1)\, dt \quad \Rightarrow \quad F'(x) = \cos 6x + 1$.

5. C p. 74

Differentiating $x + xy + 2y^2 = 6$ implicitly gives $1 + y + x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$.

Evaluating at $(2,1)$ gives: $6\frac{dy}{dx} = -2$

$$\frac{dy}{dx} = -\frac{1}{3}.$$

6. C p. 75

This limit is the definition of the derivative of $f(x) = 3x^5$ at $x = \frac{1}{2}$.

Since $f'(x) = 15x^4$, $f'(\frac{1}{2}) = 15 \cdot (\frac{1}{2})^4 = \frac{15}{16}$.

7. E p. 75

The slope of the given line $5x - y + 6 = 0$ is $m = 5$.
Since the tangent to the graph of $p(x)$ at $x = 4$ is parallel to that given line, we know that $p'(4) = 5$.

$p(x) = (x-1)(x+k) \Rightarrow p'(x) = (x-1) + (x+k)$

$\Rightarrow p'(4) = (4-1) + (4+k) = 7 + k$

Since $p'(4)$ must equal 5, we have $7 + k = 5$.
Hence $k = -2$.

---

8. A p. 75

$\cos x = e^y \Rightarrow y = \ln(\cos x)$

$\Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$

---

9. D p. 76

$a(t) = 12t^2$

$v(t) = \int a(t)\,dt = 4t^3 + C$

$v(0) = 6 \Rightarrow C = 6 \Rightarrow v(t) = 4t^3 + 6$

$s(t) = \int v(t)\,dt = t^4 + 6t + D$

$s(2) - s(0) = [16 + 12 + D] - [0 + 0 + D] = 28$

---

10. B p. 76

$A = x^2$

$\frac{dA}{dt} = 2x\frac{dx}{dt}$

Since we are given that $\frac{dA}{dt} = 3\frac{dx}{dt}$, we can substitute.

$3\frac{dx}{dt} = 2x\frac{dx}{dt} \Rightarrow x = \frac{3}{2}$

---

11. E p. 76

$$M = \frac{1}{e-1}\int_1^e \frac{1}{x}\,dx = \frac{1}{e-1}\cdot \ln e = \frac{1}{e-1}$$

---

12. B p. 77

$$\lim_{x\to\infty} \frac{3x^2+1}{(3-x)(3+x)} = \lim_{x\to\infty} \frac{3x^2+1}{9-x^2}$$

$$= \lim_{x\to\infty} \frac{3 + \frac{1}{x^2}}{\frac{9}{x^2} - 1} = -3$$

---

13. D p. 77

$$\int_{-2}^{2}(x^7+k)\,dx = \int_{-2}^{2}x^7\,dx + \int_{-2}^{2}k\,dx$$

Note that $y = x^7$ is an odd function, so $\int_{-2}^{2}x^7\,dx = 0.$

Thus $\int_{-2}^{2}(x^7+k)\,dx = 0 + 4k.$ Since we are given that the value of the definite integral is 16, we conclude that $k = 4$.

14. D p. 77

$f(x) = \frac{\tan x}{\sin x} = \frac{1}{\cos x}$ for $x \neq \pi$.

For f to be continuous at $x = \pi$, the value $f(\pi)$ must be defined in such a way so that $f(\pi) = \lim_{x\to\pi} f(x)$.

$\lim_{x\to\pi} f(x) = \lim_{x\to\pi}\frac{1}{\cos x} = \frac{1}{\cos\pi} = -1$.

15. C p. 78

$$f(x) = \frac{x^2}{1+\sin x} + e^{-2x}$$

$f'(x) = \frac{(1+\sin x)\cdot 2x - x^2\cdot(\cos x)}{(1+\sin x)^2} - 2e^{-2x}$ and $f'(0) = -2$. The function f is decreasing at $x = 0$.

16. E p. 78

$y = 3x^5 - 10x^4$
$y' = 15x^4 - 40x^3$
$y'' = 60x^3 - 120x^2$
$= 60x^2(x-2)$

The second derivative changes sign at $x = 2$, but **not** at $x = 0$. Hence the only inflection point is at $x = 2$.

17. D p. 78

Since $h(x) = f(g(x))$, we have $h'(x) = f'(g(x)) \cdot g'(x)$ by the Chain Rule.
The graph of h has a horizontal tangent line if $h'(x) = 0$.
Hence we need $f'(g(x)) \cdot g'(x) = 0$.
This occurs if $f'(g(x)) = 0$ or if $g'(x) = 0$.

Since $f'(-2)$ and $f'(1)$ both have the value 0, the first condition is satisfied if $g(x)$ is either -2 or 1.

$$g(x) = -2 \quad \text{if } x = -4 \text{ or } x = -2.$$
$$g(x) = 1 \quad \text{if } x = 0 \text{ or } x = 3.4.$$

There are 4 horizontal tangents from this condition.

The second condition is satisfied at each horizontal tangent point for the function g. These are at $x = -3$, $x = 0$, and $x = 2$.

The total list of x-values at which there are horizontal tangents is:
$x = -4, -3, -2, 0, 2, 3.4$. There are 6 places where this happens.

---

18. D p. 79

Cross sections taken perpendicular to the y-axis on the interval $[0, \frac{\pi}{2}]$ are circular. The radius of each circular cross section is an x-coordinate.
Since $y = \arcsin x$, we have $x = \sin y$.
Thus the volume is computed by $V = \pi \int_0^{\pi/2} (\sin y)^2 \, dy$.

---

19. A p. 79

$$y = xe^{-kx}$$
$$y' = e^{-kx} - kxe^{-kx} = e^{-kx}(1 - kx) \quad \Rightarrow \quad \text{Critical number at } x = \frac{1}{k}.$$

$$y\left(\frac{1}{k}\right) = \frac{1}{k} \cdot e^{-1} \quad \Rightarrow$$

The point on the curve is $\left(\frac{1}{k}, \frac{1}{ke}\right)$.

To verify that there really is a maximum value at $x = \frac{1}{k}$, use the Second Derivative Test.
$y'' = e^{-kx}(-k) - ke^{-kx}(1 - kx) = -ke^{-kx}(2 - kx)$
Then $y''\left(\frac{1}{k}\right) = -k\,e^{-1}(1)$. Since k is given to be positive, $y''\left(\frac{1}{k}\right) < 0$. Hence there <u>is</u> a maximum value at $x = \frac{1}{k}$.

20. D p. 79

The slopes of the segments in the slope field depend only upon the variable y. We can tell this because for a given y, the slopes of the segments do not change as x varies. Therefore we can eliminate the three suggested answers that have $\frac{dy}{dx}$ depend upon x. For $y = 0$, the slopes are positive. The correct answer cannot be $\frac{dy}{dx} = y - 2y^2$, for that would have a slope of 0 when $y = 0$.

---

21. B p. 80

$y' > 0 \Rightarrow$ y is an increasing function.
$y'' < 0 \Rightarrow$ y is concave down.

The only increasing and concave down curve is (B).

---

22. D p. 80

Since the inputs are not equally spaced, we must calculate the area of the individual trapezoids.

$$\frac{1}{2}(4)[3 + k] + \frac{1}{2}(3)[k + 9] + \frac{1}{2}(2)[9 + 11] = 57$$

$$(6 + 2k) + (\frac{3}{2}k + \frac{27}{2}) + (20) = 57$$

$$12 + 4k + 3k + 27 + 40 = 114$$

$$7k = 35$$

$$k = 5$$

---

23. C p. 80

$$f(x) = \frac{x^2 + 10}{e^x} = (x^2 + 1)e^{-x}$$

$$f'(x) = 2xe^{-x} - (x^2 + 1)e^{-x}$$

$$= \frac{2x - x^2 - 1}{e^x} = -\frac{(x-1)^2}{e^x}.$$ Thus, $f'(x) < 0$ for all $x \neq 1$.

Hence, f is decreasing everywhere, except when $x = 1$.

$$f''(x) = (2 - 2x)e^{-x} + (2x - x^2 - 1)(-e^{-x}) = \frac{2 - 2x - 2x + x^2 + 1}{e^x} = \frac{x^2 - 4x + 3}{e^x}$$

$$= \frac{(x-3)(x-1)}{e^x}$$

f is concave down when $f''(x) < 0$; that is, when $1 < x < 3$.

24. A p. 81

Since the rate of growth is $1500\,e^{3t/4}$, we start with $\frac{dx}{dt} = 1500\,e^{3t/4}$.

Then $dx = 1500\,e^{3t/4}\,dt$.

$x = 2000\,e^{3t/4} + C$

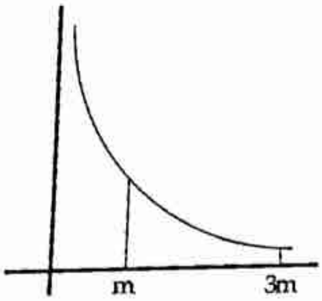
Knowing that $x = 2000$ when $t = 0$ allows us to evaluate C:

$2000 = 2000\,e^{0} + C \quad \Rightarrow \quad C = 0.$

Hence $x = 2000\,e^{3t/4}$.

When $t = 4$, we have $x = 2000\,e^{3}$.

25. A p. 81



$$\int_m^{3m} \frac{1}{x}\,dx = \ln|x|\Big]_m^{3m} = \ln(3m) - \ln m = \ln 3.$$

This result is independent of m.

26. C p. 82

$$x(t) = \ln t + \frac{t^2}{18} + 1 \quad \Rightarrow \quad v(t) = x'(t) = \frac{1}{t} + \frac{t}{9}$$
$$\Rightarrow \quad a(t) = x'(t) = -\frac{1}{t^2} + \frac{1}{9}.$$

The acceleration is zero when $t = 3$.

Then $v(3) = \frac{1}{3} + \frac{3}{9} = \frac{2}{3}$.

27. D p. 82

$$\int 6 \sin x \cos^2 x\,dx = -6\int (\cos x)^2 (-\sin x)\,dx$$
$$= -6\frac{\cos^3 x}{3} + C = -2\cos^3 x + C$$

28. D p. 82

By the Second Fundamental Theorem, $G'(x) = \sin(\ln 2x)$.

Then $G''(x) = \cos(\ln 2x) \cdot \frac{1}{2x} \cdot 2 = \frac{\cos(\ln 2x)}{x}$.

Hence $G''(\frac{1}{2}) = \frac{\cos(\ln 1)}{1/2} = 2\cos 0 = 2.$

Exam IV
Section I
Part B — Calculators Permitted

1. D p. 83

I. $f''(x)$ changes sign at $x = -1$. True

II. $f''(x) < 0$ on the interval $(-1,3)$. True

III. Since $\frac{d(f'(x))}{dx} < 0$ in the vicinity of $x = 1$, the function f' is decreasing at $x = 1$. False

---

2. D p. 83

$$\frac{\ln x^2 - x\ln x}{x-2} = \frac{2\ln x - x\ln x}{x-2} = \frac{(2-x)\ln x}{x-2} = -\ln x \text{ for } x \neq 2$$

Since f is given to be continuous at $x = 2$, $\lim_{x\to 2} f(x) = f(2)$.

But $\lim_{x\to 2} f(x) = -\ln 2$. Hence $f(2) = -\ln 2$.

---

3. E p. 84

Condition		Interpretation	Points
$f(x) > 0$	$\Rightarrow$	the point $(x,f(x))$ is above the x-axis.	M, P, Q, R
$f'(x) < 0$	$\Rightarrow$	f is decreasing.	M, R
$f''(x) < 0$	$\Rightarrow$	the graph of f is concave down.	Q, R

All **three** conditions occur only at point R.

---

4. D p. 84

With the substitution $u = \sqrt{1+x}$, we have $x = u^2 - 1$ and $dx = 2u\,du$.

Then $\int 60x\sqrt{1+x}\,dx = \int 60(u^2-1)\cdot u\cdot 2u\,du$

$= \int (120u^4 - 120u^2)\,du = 24u^5 - 40u^3 + C$

---

5. B p. 84

$$f'(x) = \frac{2x}{2\sqrt{x^2 + .0001}}.$$

I. Since $f'(0)$ exists, f is continuous at $x = 0$. False

II. $f'(0) = 0$. Thus there is a horizontal tangent at $x = 0$. True

III. $f'(x)$ is defined as above with $f'(0) = 0$. False

6. C p. 85

$f(x) = \frac{1}{x^2} = x^{-2}$ $g(x) = \arctan x$

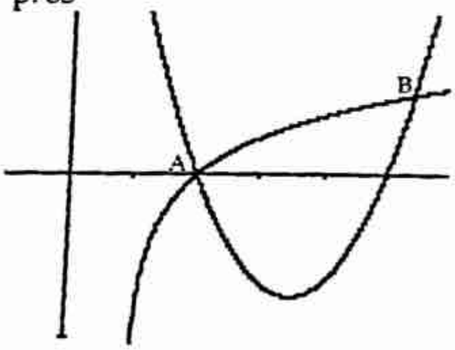
$f'(x) = -\frac{2}{x^3}$ $g'(x) = \frac{1}{1+x^2}$

When $\frac{1}{1+x^2} = \frac{-2}{x^3}$, then $x^3 = -2 - 2x^2$.

Solving $x^3 + 2x^2 + 2 = 0$ gives $x = -2.359$.

---

7. B p. 85



The curves intersect at

A, where $x = 2$

and B, where $x = 5.4337$.

Then the area is:

$$A = \int_2^{5.4337} (g(x) - f(x))\,dx \approx 7.36.$$

---

8. B p. 85

The average rate of change of a function f over an interval $[a, b]$ is defined to be $\frac{f(b) - f(a)}{b - a}$.

With the function $f(x) = \int_0^x \sqrt{1 + \cos(t^2)}\,dt$ and the interval $[1, 3]$,

we have, first of all, $f(3) - f(1) = \int_0^3 \sqrt{1 + \cos(t^2)}\,dt - \int_0^1 \sqrt{1 + \cos(t^2)}\,dt$

$$= \int_1^3 \sqrt{1 + \cos(t^2)}\,dt.$$

Hence the average rate of change of the function is $\frac{1}{2}\int_1^3 \sqrt{1 + \cos(t^2)}\,dt \approx 0.86$.

---

9. D p. 86

I. $f'(0) = 1 \Rightarrow$ f is increasing at $x = 1$. False

II. $f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$.
Hence f is increasing to the left of $x = 2$ and decreasing to the right of $x = 2$. There is a relative maximum there. True

III. $f'(x)$ is increasing on an open interval containing $x = -1$.
Hence, the graph is concave up at $x = -1$. True

10. D p. 86

$$f(x) = 3 + \int_2^x \frac{20}{1+t^2}\,dt \qquad f(2) = 3 + \int_2^2 \frac{20}{1+t^2}\,dt = 3 + 0 = 3$$

$$f'(x) = \frac{20}{1+x^2} \qquad f'(2) = \frac{20}{5} = 4$$

The equation of the tangent line is: $y - 3 = 4(x - 2)$.

---

11. C p. 86
Parts of the rectangles are above the curves for B and C.
The trapezoids are all on or under the curves for A, C, and E.
Hence, the answer is C.

---

12. B p. 87

$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

At $t = 6$ seconds, $\frac{dV}{dt}$ can be estimated from the graph. The slope of the tangent to the curve at $t = 6$ is approximately $\frac{2\pi}{2} = \pi$ in^3/sec.

Hence we use $\frac{dV}{dt} = \pi$.

In addition, when $t = 6$, $V = 4\pi$. This allows us to find the radius r when $t = 6$. From the original volume formula,

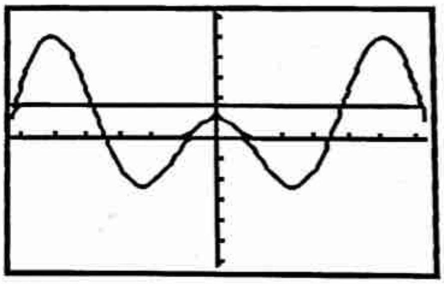
$\frac{4}{3}\pi r^3 = 4\pi \quad \Rightarrow \quad r^3 = 3 \quad \Rightarrow \quad r = 3^{1/3}$.

Now from the formula for the rate of change of the volume, we have:

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \Rightarrow \quad \pi = 4\pi(3^{1/3})^2 \frac{dr}{dt}$. Hence $\frac{dr}{dt} = \frac{1}{4 \cdot 3^{2/3}} \approx 0.12$

---

13. B p. 87



$y = x\cos x$
$y' = \cos x - x\sin x$

We want $\cos x - x\sin x = \frac{\pi}{2}$.

Shown to the right are graphs of

$Y_1 = \cos x - x\sin x$

and $Y_2 = \frac{\pi}{2}$.

The viewing window is:

$-2\pi \le x \le 2\pi$; $-6.2 \le y \le 6.2$.

Y_1 and Y_2 intersect four times on the interval $[-2\pi, 2\pi]$.

14. C p. 88

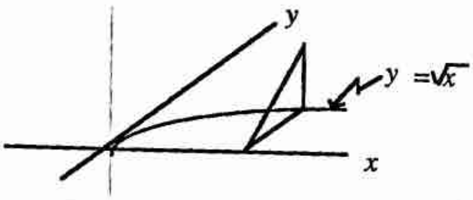
$f(6) - f(2) = \int_2^6 f'(t)\,dt$

Then $f(6) = f(2) + \int_2^6 f'(x)\,dx$ (area of quarter circle = $\frac{\pi r^2}{4}$)

$= 3 + \frac{1}{4}\pi(4)^2 = 3 + 4\pi$

15. B p. 88

Since the cross sections perpendicular to the x-axis are isosceles right triangles, we want to create an integral of the form $\int_a^b f(x)\,dx$ for this volume.



A leg of the triangle is a typical y-coordinate and the cross sectional area is $\frac{1}{2}yy = \frac{1}{2}y^2$ where $y = \sqrt{x}$. Hence, the volume is $V = \int_0^4 \frac{1}{2}x\,dx = \frac{x^2}{4}\Big|_0^4 = 4$.

16. D p. 89

Since the graph of f is made up of straight line segments and a semicircle, we can evaluate F at each of the integer coordinates from x = 0 through x = 8. Note that sections of the graph below the t-axis decrease the value of F.

x	0	1	2	3	4	5	6	7	8
F(x)	0	.25	1	1.5	.715	−.071	.429	.929	.429

The value of F(x) changes from positive to negative at an x-coordinate between 4 and 5.
The value of F(x) changes from negative to positive at an x-coordinate between 5 and 6.
Since $f(t) \geq 0$ for $0 \leq t \leq 3$ and for $5 \leq t \leq 7$, F is increasing on those intervals.
Since $f(t) \leq 0$ for $3 \leq t \leq 5$ and for $7 \leq t \leq 8$, F is decreasing on those intervals.
Hence the only zeros for the function F occur in the intervals [4,5] and [5,6].

17. B p. 89

Separate variables to solve the differential equation.

$\frac{dN}{dt} = 2N \Rightarrow \frac{dN}{N} = 2dt$

$\Rightarrow \ln N = 2t + C$

$\Rightarrow N = e^{2t+C} = e^{2t} \cdot e^{C} = De^{2t}$ (where $D = e^{C}$)

Since $N = 3$ when $t = 0$, we have $3 = De^0$, which implies $D = 3$.
Thus we can write: $N = 3e^{2t}$. Now let $N = 1210$.
Then we have $1210 = 3e^{2t} \Rightarrow \frac{1210}{3} = e^{2t}$

$\Rightarrow 2t = \ln\frac{1210}{3}$

$\Rightarrow t = \frac{\ln 1210 - \ln 3}{2} \approx 3$

Exam IV
Section II
Part A — Calculators Permitted

1. p. 91

(a) Speed is greatest at $t = 8$ since that is when the magnitude of the velocity is the greatest. — 1: answer

(b) Particle changes direction at $t = 4$ and $t = 11$ because the velocity changes sign at each of these points in time. — $2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c) The total distance traveled by the particle is the sum of the areas of the regions between the velocity graph and the x-axis.

$$\int_0^{12} |v(t)|\,dt = 2\pi + \left|-\frac{21}{2}\right| + \frac{1}{2} = 2\pi + 11$$

$3: \begin{cases} 2: \text{equation} \\ 1: \text{answer} \end{cases}$

(d) Using the Fundamental Theorem of Calculus :

$$x(8) = x(0) + \int_0^8 v(t)\,dt = 2 + (2\pi - 6) = 2\pi - 4$$

$3: \begin{cases} 1: \text{handles initial condition} \\ 1: \int_0^8 v(t)\,dt \\ 1: \text{answer} \end{cases}$

2. p. 92

(a) $\int_0^{10} f(t)\,dt = 482.954$ — $2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) At time t = 6, crude oil is being pumped into the tank at a rate of f(6) = 37.2 barrels per hour and removed at a constant rate of 50 barrels per hour. Thus,with 37.2 < 50, the level of oil in the tank is falling at $t = 6$.

$3: \begin{cases} 1: f(6) \\ 1: \text{compares } f(6) \text{ to } 50 \\ 1: \text{conclusion} \end{cases}$

(c) At time t = 10, the tank contains $300 + \int_0^{10}(f(t) - 50)\,dt = 282.954$ barrels of crude oil. — 1: answer

(d) The absolute minimum occurs at a critical point or at an endpoint. If $F(t) = 300 + \int_0^t (f(x) - 50)\,dx$ represents the volume of crude oil in the tank at time t, then $F'(t) = f(t) - 50$ and $f(t) - 50 = 0$ at inputs $t = 4.574$ and $t = 9.268$.

At critical points: $F(4.574) = 336.052$
$F(9.278) = 279.374$;

At endpoints: $F(0) = 0$
$F(10) = 282.954$

The minimum number of barrels of crude oil is 279.374 barrels at $t = 9.278$ hrs

$3: \begin{cases} 1: \text{considers } F'(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

Exam IV
Section II
Part B — No Calculators

3. p. 93

(a) Left-hand Riemann Sum = $(1)(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}) = \frac{25}{12}$

1: answer

(b) $A = \int_0^4 \frac{1}{x+1}\,dx = \ln|x+1|\Big]_0^4 = \ln 5 - \ln 1 = \ln 5$

2: { 1: integrand; 1: answer }

(c) $\int_0^k \frac{1}{x+1}\,dx = \int_k^4 \frac{1}{x+1}\,dx$

$\ln|x+1|\Big]_0^k = \ln|x+1|\Big]_k^4$

$\ln(k+1) = \ln(5) - \ln(k+1)$ $\quad [k+1>0 \Rightarrow \ln|k+1| = \ln(k+1)]$

$2\ln(k+1) = \ln(5)$

$\ln(k+1) = \ln(\sqrt{5})$

$k = \sqrt{5} - 1$

3: { 1: equation; 1: antiderivatives; 1: answer }

(d) Length of a side $= \frac{1}{x+1}$

Area of an equilateral triangle $= \frac{\sqrt{3}}{4}(\text{side})^2$

Area of a cross section $= \frac{\sqrt{3}}{4}\left(\frac{1}{x+1}\right)^2$

Volume $= \frac{\sqrt{3}}{4}\int_0^4 \left(\frac{1}{x+1}\right)^2 dx$

3: { 1: limits; 1: integrand; 1: answer }

4. p. 94

(a) The absolute maximum and absolute minimum values occur at either a critical point or at an endpoint. The first derivative $f'(x) = 1 + 2\sin x$ has zeros at $x = 7\pi/6$ and $x = 11\pi/6$. At these critical points we have $f(7\pi/6) = 7\pi/6 + \sqrt{3} \approx 5$ and $f(11\pi/6) = 11\pi/6 - \sqrt{3} \approx 4$. At the endpoints we have $f(0) = -2$ and $f(2\pi) = 2\pi - 2 \approx 4$. Therefore, on the interval $[0, 2\pi]$, the function's absolute maximum value, $7\pi/6 + \sqrt{3}$, occurs at $x = 7\pi/6$ and its absolute minimum value, -2, occurs at $x = 0$.

4: { 3: identifies critical numbers and endpts as candidates; 1: answer }

q

(b) $f''(x) = 0$ on $[0, 2\pi]$ when $\cos x = 0$ and $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

$f''(x) < 0$ when $\frac{\pi}{2} < x < \frac{3\pi}{2}$. Hence, f is concave down on $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

2: { 1: answer; 1: justification }

(c) $$y_{ave} = \frac{1}{2\pi - 0}\int_0^{2\pi}(x - 2\cos x)\,dx = \frac{1}{2\pi}\left(\frac{x^2}{2} - 2\sin x\right)_0^{2\pi}$$

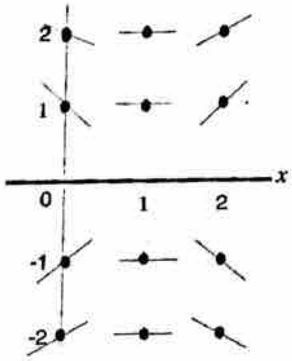
$$= \frac{1}{2\pi}\left(\left(\frac{4\pi^2}{2} - 0\right) - (0 - 0)\right) = \pi.$$

3: { 1: limits and average value constant; 1: antiderivative; 1: answer }

5. p. 95

(a) We calculate slopes at each of the twelve points.

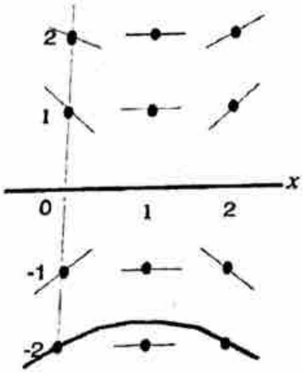
At (0,1), m = –1	At (0,–1), m = 1.
At (0,2), m = –1/2.	At (0,–2), m = 1/2.
At (1,1), m = 0.	At (1,–1), m = 0.
At (1,2), m = 0.	At (1,–2), m = 0.
At (2,1), m = 1.	At (2,–1), m = –1.
At (2,2), m = 1/2.	At (2,–1), m = –1/2.



Then draw short line segments through each of the points with the appropriate slope.

$2: \begin{cases} 1\text{: zero slope} \\ 1\text{: nonzero slope} \end{cases}$

(b)



1 : solution curve must go through (0 ,–2), and follow the given slope lines and extend to the border

(c) $y''(x) = \dfrac{y \cdot 1 - (x-1) \cdot y'}{y^2}$

$= \dfrac{y \cdot 1 - (x-1)\cdot\left(\frac{x-1}{y}\right)}{y^2}$

$= \dfrac{y^2 - (x-1)^2}{y^3}.$

At $(0,-2)$, $y''(0) = \dfrac{4-(-1)^2}{(-2)^3} = -\dfrac{3}{8}$

$2: \begin{cases} 1\text{: } y''(x) \\ 1\text{: answer} \end{cases}$

(d) $\int y \dfrac{dy}{dx} = \int (x-1)\, dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} - x + C$

$y(0) = -2 \Rightarrow C = 2$

$y^2 = x^2 - 2x + 4$

$|y| = \sqrt{x^2 - 2x + 4}$. Since the initial condition has $y < 0$,

choose $y = -\sqrt{x^2 - 2x + 4}$.

$4: \begin{cases} 1\text{: separates variables} \\ 1\text{: antiderivatives} \\ 1\text{: uses initial conditions} \\ 1\text{: solves for } y \end{cases}$

6. p. 96

(a) $G(x) = \int_{-4}^{x} f(t)\,dt$

1 : answer

Then $G(-4) = \int_{-4}^{-4} f(t)\,dt$. Provided the function f is defined at $t = -4$ (and it is), then this integral has a value of 0.

(b) By the Second Fundamental Theorem, $G'(x) = f(x)$.

Thus $G'(-1) = f(-1) = 2$.

$2: \begin{cases} 1: G'(x) = f(x) \\ 1: \text{answer} \end{cases}$

(c) The given graph is the derivative of G. The graph of G will be concave down if its derivative is decreasing; that is, if its second derivative is negative-valued. This occurs on the open intervals $(-4,-3)$ and $(-1,2)$.

$3: \begin{cases} 2: \text{intervals} \\ 1: \text{justification} \end{cases}$

(d) The maximum value of G occurs at either a critical point or an endpoint of the interval. G has critical points at $x = 1$ and $x = 3$, where $G'(x) = f(x) = 0$. By summing areas of regions we estimate that at the critical points

$$G(1) = \int_{-4}^{1} f(x)\,dx = 7 \quad \text{and} \quad G(3) = 7 - \frac{4}{3} = \frac{17}{3}.$$

$3: \begin{cases} 1: \text{answer} \\ 2: \text{justification} \end{cases}$

At the endpoints,

$$G(-4) = \int_{-4}^{-4} f(x)\,dx = 0 \text{ and } G(4) = \int_{-4}^{4} f(x)\,dx = 7 - \frac{4}{3} + 1 = \frac{20}{3}$$

Therefore, G has its maximum value of 7 at $x = 1$.

Exam V
Section I
Part A — No Calculators

1. B p. 97

$y = \cos^2(2x)$

$\frac{dy}{dx} = 2\cos(2x)(-\sin(2x)) \cdot 2 = -4\sin(2x)\cos(2x)$

2. C p. 97

I.	The units on the axes are equal. At (2,2), the slope ≈ 1.	True
II.	As y-coordinates are approaching 8 (from above and below), the slope lines are flattening out.	True
III.	For a given value of x, the slopes vary at different heights.	False

3. C p. 98

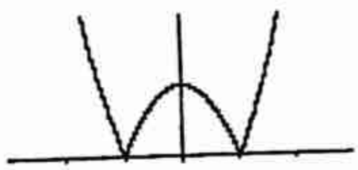
$y = \ln\sqrt{x} = \ln x^{1/2} = \frac{1}{2}\ln x$

Then $\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x}$

At $(e^2, 1)$, $\frac{dy}{dx} = \frac{1}{2e^2}$

4. C p. 98

$f(x) = |x^2 - 1|$ is not differentiable at $x = \pm 1$.
The graph has sharp corners there.



$f(x) = \sqrt{x^2 - 1}$ is not continuous for x between −1 and 1 because it is undefined there.

$f(x) = \sqrt{x^2 + 1}$ is differentiable for all real x; $f'(x) = \frac{2x}{2\sqrt{x^2+1}}$. Hence the function is continuous for all real x.

$f(x) = \frac{1}{x^2 - 1}$ is not continuous at $x = \pm 1$ since the function is undefined there.

The third function (C) is both continuous and differentiable.

5. D p. 98

The slope of the line though (9,3) and (1,1) is $m = \frac{3-1}{9-1} = \frac{1}{4}$.

Since $y = \sqrt{x}$, we have $y' = \frac{1}{2\sqrt{x}}$

Since the tangent line is to have slope m, we have $\frac{1}{2\sqrt{x}} = \frac{1}{4} \Rightarrow x = 4$.

6. A p. 99

$f(x) = \frac{x^4}{2} - \frac{x^5}{10}$

$f'(x) = 2x^3 - \frac{1}{2}x^4.$

To maximize $f'(x)$, take $f''(x) = 6x^2 - 2x^3 = 2x^2(3-x)$.

$f'(x)$ is an upside-down quartic, so has a maximum. Critical numbers for $f'(x)$ are the zeros of $f''(x)$; they are $x = 0$ and $x = 3$. $f'(x)$ is increasing on either side of $x = 0$, so that is **NOT** the maximum. $f''(x) > 0$ when $x < 3$ and $f''(x) < 0$ when $x > 3$. Therefore, f' attains its maximum at $x = 3$.

---

7. E p. 99

$a(t) = 4 - 6t$

$v(t) = \int a(t)\,dt = 4t - 3t^2 + C$

$v(0) = 20 \Rightarrow C = 20 \Rightarrow v(t) = 4t - 3t^2 + 20$

$s(t) = \int v(t)\,dt = 2t^2 - t^3 + 20t + D$

$s(3) - s(1) = (18 - 27 + 60 + D) - (2 - 1 + 20 + D) = (51 + D) - (21 + D) = 30$

---

8. D p. 99

$$\lim_{x\to 1} \frac{\sqrt{x+3} - 2}{1-x} = \lim_{x\to 1} \frac{[\sqrt{x+3} - 2][\sqrt{x+3} + 2]}{(1-x)[\sqrt{x+3} + 2]}$$

$$= \lim_{x\to 1} \frac{(x+3) - 4}{(1-x)[\sqrt{x+3} + 2]}$$

$$= \lim_{x\to 1} \frac{x-1}{(1-x)[\sqrt{x+3} + 2}$$

$$= \lim_{x\to 1} \frac{-1}{\sqrt{x+3} + 2} = -\frac{1}{4} = -0.25$$

---

9. A p. 100

To achieve continuity at $x = 1$ (the only place in question), we need $\lim_{x\to 1} f(x) = f(1)$.

But $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2 - 2x + 1}{x - 1} = \lim_{x\to 1} (x-1) = 0$ while $f(1) = k$.

Therefore $k = 0$.

---

10. A p. 100

The average value of $f(x)$ on $[a,b]$ is defined to be $\frac{1}{b-a}\int_a^b f(x)\,dx$. Therefore

$$M = \frac{1}{\frac{1}{2} - 0} \cdot \int_0^{1/2} (e^{2x} + 1)\,dx = 2\left[\frac{1}{2}e^{2x} + x\right]_0^{1/2} = 2\left[\left(\frac{1}{2}e + \frac{1}{2}\right) - \left(\frac{1}{2} + 0\right)\right] = e$$

---

11. B p. 100

$v(t) = \frac{e^t}{t}$

$v'(t) = \frac{te^t - e^t}{t^2} = \frac{e^t(t-1)}{t^2}$, so the only critical number is $t = 1$.

If $0 < t < 1$, then $v'(t) < 0$, so v is decreasing.
If $t > 1$, then $v'(t) > 0$, so v is increasing.
Thus v achieves its minimum value at $t = 1$.

---

12. D p. 101

$$\int \frac{4x}{1+x^2}\,dx = 2\int \frac{2x}{1+x^2}\,dx = 2\ln(1+x^2) + C$$

(Note that $1 + x^2$ is always positive-valued.)

---

13. D p. 101

$f(x) = x^4 + ax^2 + b$	$f(0) = 1$	$\Rightarrow$	$1 = b$
$f'(x) = 4x^3 + 2ax$	$f'(0) = 0$	$\Rightarrow$	$0 = 0$
$f''(x) = 12x^2 + 2a$	$f''(1) = 0$	$\Rightarrow$	$0 = 12 + 2a$
	Thus $a = -6$; $b = 1$.		

---

14. A p. 101

$$\int_1^2 \frac{x^2 - x}{x^3}\,dx = \int_1^2 (x^{-1} - x^{-2})\,dx = \left[\ln|x| + \frac{1}{x}\right]_1^2$$

$$= \left(\ln 2 + \frac{1}{2}\right) - (\ln 1 + 1) = \ln 2 - \frac{1}{2}$$

---

15. C p. 102

Let x denote the edge of the cube. Then

$V = x^3$ $\qquad$ $SA = 6x^2$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$ $\qquad$ When $SA = 150$, then $x = 5$.

We are given that $\frac{dx}{dt} = 0.2$. Hence $\frac{dV}{dt} = 3 \cdot 5^2 \cdot (0.2) = 15$

16. B p. 102

$$\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{1+x^2}} = \int_0^{\sqrt{3}} \frac{2x}{2\sqrt{1+x^2}}dx = \sqrt{1+x^2}\,\Big]_0^{\sqrt{3}} = 2-1 = 1$$

This can also be done with a formal substitution.

Let $u = x^2 + 1$, so that $du = 2x\,dx$, and $x\,dx = \frac{1}{2}du$.

To change the limits of integration, we note that $x = 0 \Rightarrow u=1$
and $x = \sqrt{3} \Rightarrow u=4$.

Then $\displaystyle\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{1+x^2}} = \int_1^4 \frac{\frac{1}{2}du}{u^{1/2}} = \frac{1}{2}\int_1^4 u^{-1/2}\,du = u^{1/2}\Big]_1^4 = 2-1 = 1$

---

17. D p. 102

Solution I. Work analytically.

If $f(x) = \ln|x^2-4|$ on the interval $(-2,2)$,

then $f'(x) = \dfrac{2x}{x^2-4} = \dfrac{2x}{(x-2)(x+2)}$.

(A) $f'(x) < 0$ when $0 < x < 2$; f is decreasing then. **False.**

(B) $f(0) = \ln 4 \neq 0$, so $(0,0)$ is **not** on the graph. **False.**

(C) $|x^2-4|$ has a maximum value of 4 on the given domain, so $f(x)$ has a maximum value of $\ln(4)$. **False.**

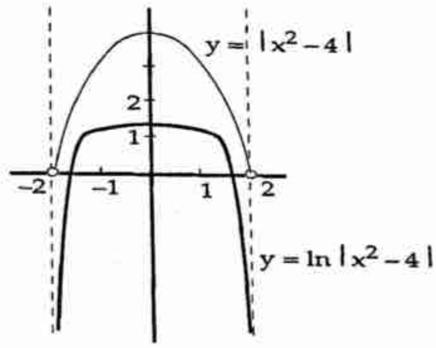
(E) Since $f(0) = \ln 4$, f does **not** have an asymptote at $x = 0$. **False.**

Thus (D) must be **True**.

Solution II. Work graphically.

The graph of $y = |x^2-4|$, restricted to the open interval $(-2,2)$, is the tip of an upside-down parabola, as shown to the right.

$f(x) = \ln(y) = \ln|x^2-4|$ then has the darker graph shown. The selection of the correct answer is made easier with these graphs.



---

18. E p. 103

$g(x) = \text{Arcsin}(2x)$

$$g'(x) = \frac{1}{\sqrt{1-(2x)^2}}\cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

19. B p. 103

$$\int x(x^2-1)^4\,dx = \frac{1}{2}\int (x^2-1)^4\,(2x\,dx)$$
$$= \frac{1}{2}\cdot\frac{(x^2-1)^5}{5} + C = \frac{1}{10}(x^2-1)^5 + C$$

20. B p. 103

$y = e^{kx}$ $\qquad \frac{d^3y}{dx^3} = k^3e^{kx}$

$\frac{dy}{dx} = ke^{kx}$ $\qquad \frac{d^4y}{dx^4} = k^4e^{kx}$

$\frac{d^2y}{dx^2} = k^2e^{kx}$ $\qquad \frac{d^5y}{dx^5} = k^5e^{kx}$

21. B p. 104

I. $f(2) = 1;\ f'(1) = 1.$ False

II. $\int_0^1 f(x)\,dx = -\frac{1}{2};$ $\qquad f'(3.5) = -1$ True

III. $\int_{-1}^1 f(x)\,dx = -1;$ $\qquad \int_{-1}^2 f(x)\,dx = \frac{1}{2}$ False

22. C p. 104

$g(x) = \sqrt{x}\,(x-1)^{2/3}$
$g'(x) = \frac{1}{2\sqrt{x}}(x-1)^{2/3} + \frac{2}{3}(x-1)^{-1/3}\sqrt{x}$

x must be positive (so that $\frac{1}{2\sqrt{x}}$ exists).

x must not equal 1 (so that $(x-1)^{-1/3}$ exists).

The domain is $\{\,x \mid x > 0 \text{ and } x \neq 1\,\} = \{\,x \mid 0 < x < 1 \text{ or } x > 1\,\}$.

23. D p. 104

$x(t) = 10t - 4t^2$
$x'(t) = 10 - 8t$

$x'(t) > 0$ if $t < \frac{5}{4}$. The point is moving to the right.

$x'(t) < 0$ if $t > \frac{5}{4}$. The point is moving to the left.

The total distance traveled is $\left[x(\tfrac{5}{4}) - x(1)\right] + \left[x(\tfrac{5}{4}) - x(2)\right]$.

Since $x(\frac{5}{4}) = \frac{25}{4}$, $x(1) = 6$, and $x(2) = 4$, the total distance traveled is $\frac{5}{2}$.

24. D p. 105

$g'(x) = 0 \Rightarrow$ a horizontal tangent at x.
$g''(x) = 0 \Rightarrow$ no concavity (with possibly a change in concavity).

Only at points B, D, and E is there a horizontal tangent to the graph of g, so $g'(0) = 0$ at those points.

At B, the graph is concave up, so $g''(x) > 0$ there.
At D, the concavity changes, so $g''(x) = 0$ there.
At E, the graph is concave down, so $g''(x) < 0$ there.
Hence, point D is the answer.

---

25. B p. 105

$xy - 2y + 4y^2 = 6$
$x\frac{dy}{dx} + y - 2\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$
$\frac{dy}{dx} = \frac{-y}{x + 8y - 2}$

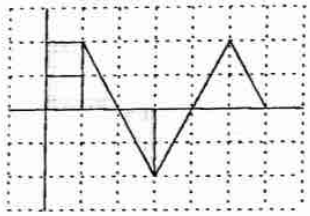
When $y = 1$,
$x - 2 + 4 = 6$;
$\Rightarrow x = 4$.

Hence at the point (4,1), $\frac{dy}{dx} = \frac{-1}{4+8-2} = -\frac{1}{10}$

---

26. D p. 105

$G'(x) = f(x) \Rightarrow G'(3) = -2$.
G(3) is evaluated by counting up areas. The region on [2,3] whose negative-valued integral cancels out the region on [1,2] with the positive-valued integral. Hence $G(3) = 2$. Then a linear approximation for G near $x = 2$ is given by $y - 2 = -2(x - 3)$. Hence $y = -2x + 8$.



---

27. A p. 106

Separate variables to solve this differential equation.

$\frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = -2xy$
$\Rightarrow \frac{dy}{y} = -2x\,dx$
$\Rightarrow \ln|y| = -x^2 + C$

Since the curve contains the point (0, e), we have $\ln e = C$, so $C = 1$.
Thus $\ln y = -x^2 + 1$, or $y = e^{1-x^2}$.

---

28. A p. 106

I. $F(1) = \int_1^1 \ln(2t-1)\,dt = 0$ since $\int_a^a f(t)\,dt = 0$ if f(a) exists. True

II. $F'(x) = \ln(2x-1)$ by the Second Fundamental Theorem.
Then $F'(1) = \ln(1) = 0$. True

III. $F''(x) = \frac{2}{2x-1}$, so $F''(1) = 2$. False

Exam V
Section I
Part B — Calculators Permitted

1. C p. 107

$f(x) = e^x(x^2+1)$

$f'(x) = e^x \cdot 2x + (x^2+1)e^x = e^x(x^2+2x+1)$

$f''(x) = e^x(2x+2) + (x^2+2x+1)e^x = e^x(x^2+4x+3) = e^x(x^2+4x+3)$

$= e^x(x+1)(x+3) \qquad \{-3, -1\}$

The graph of f has points of inflection at $x = -3$ and $x = -1$. (f'' changes from positive to negative at $x = -3$, meaning that the graph changes from concave up to concave down; at $x = -1$, the concavity switches back again.) Thus, there are 2 points of inflection. The answer is C.

2. C p. 107

$$\text{Volume} = \int_0^{10} 500\, e^{-.2t}\, dt = \frac{500}{-.2}\left[e^{-.2t}\right]_0^{10} \approx 2161.7$$

3. B p. 108

Given the sales function S(t), the rate of change of sales is given by
$S'(t) = -.46\,(.45)\sin(.45t + 3.15)$. The derivative of S' is
$S''(t) = -.46\,(.45)(.45)\cos(.45t + 3.15)$.

Using a calculator, you find that the function S'' has one zero on the interval [0, 10] at $t = 3.4719$. Since $S''(t) > 0$ for $0 < t < 3.4719$ and $S''(t) < 0$ for $3.4719 < t < 10$, the function S' has a maximum value at $t = 3.4719$.

This corresponds to a point in 1983.

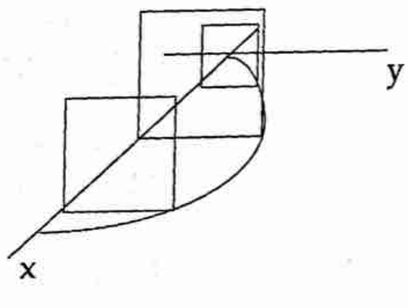
4. E p. 108

The midpoint approximation is given by

$M_4 = 2\left[f(2) + f(4) + f(6) + f(8) \right]$
$= 2\left[2+4+3+3\right] = 24$

5. D p. 108

The interval for one arch of the graph of $y = \sin x$ is $[0,\pi]$.
Since the regular cross sections are squares with edge $y = \sin x$, the volume of the solid is



$$V = \int_0^{\pi} (\sin x)^2\, dx \approx 1.57.$$

6. C p. 109

$g(f(x)) = x \quad \Rightarrow \quad g'(f(x))\,f'(x) = 1$

$\Rightarrow \quad g'(f(x)) = \frac{1}{f'(x)}$

To find $g'(2)$, first determine the number c such that $f(c) = 2$. Then we compute $\frac{1}{f'(c)}$.

To find the number c such that $f(c) = 2$, use your calculator to determine the zero of the function $F(x) = 2x + \sin x - 2$. It is $c = 0.684$.

Then $g'(f(.684)) = g'(2) = \frac{1}{f'(.684)} = \frac{1}{2 + \cos .684} = 0.36$.

---

7. D p. 109

$E(B) = 14000 + (B+1)^2$ and $B(t) = 20 \sin \frac{t}{10} + 50$.

$\frac{dE}{dt} = \frac{dE}{dB} \cdot \frac{dB}{dt}$

$= 2(B+1) \cdot 2 \cos \frac{t}{10} = 4(B+1) \cos \frac{t}{10}$

When $t = 100$, then $B = 20 \sin 10 + 50 \approx 39.12$.

Hence $\frac{dE}{dt} \approx -134.65$. Thus E is decreasing at $135 per day.

---

8. B p. 110

I.	f is increasing to the left of -1 and decreasing to the right.	True
II.	$f'(0) < 0$, so f is decreasing at $x = 0$.	False
III.	f' is decreasing on $(-2,0)$, so $f''(x) < 0$ on that interval.	True
IV.	f' has horizontal tangents at $x = 0, 2$, and 3, and $f''(x)$ changes sign at each point.	True

---

9. E p. 110

To have the graph of f both concave down and increasing, we must have $f''(x) < 0$ <u>and</u> $f'(x) > 0$.

$f(x) = 4x^{3/2} - 3x^2$

$f'(x) = 6x^{1/2} - 6x = 6x^{1/2}(1 - x^{1/2})$

$f''(x) = 3x^{-1/2} - 6 = 3x^{-1/2}(1 - 2x^{1/2})$

Note that the presence of the factor $x^{-1/2}$ in $f''(x)$ forces $x > 0$, since $f''(x)$ must be defined in order to be negative.

Also $x^{1/2}$ is always positive, so it is the other factors we must deal with in determining whether $f'(x)$ and $f''(x)$ are positive or negative.

In order to have	$f'(x) > 0$	and	$f''(x) < 0$,
we must have	$1 - x^{1/2} > 0$	and	$1 - 2x^{1/2} < 0$.
Hence	$x^{1/2} < 1$	and	$\frac{1}{2} < x^{1/2}$.
Therefore	$0 < x < 1$	and	$\frac{1}{4} < x$

This interval is $(\frac{1}{4}, 1)$.

10. B p. 110

$f(x) = x^2 - \frac{1}{e^x}$

$y_{inst} = f'(x) = 2x + \frac{1}{e^x}$

$y_{ave} = \frac{f(3) - f(0)}{3 - 0} = \frac{\left(9 - \frac{1}{e^3}\right)}{3} = \frac{10 - \frac{1}{e^3}}{3} = 3.3167$

Solving $2x + \frac{1}{e^x} = 3.3167$ gives $x = 1.5525$.

---

11. E p. 111

Solution I. Differentiate the given equation: $3\cos(3x) = f(x)$.

$\int_a^x 3\cos(3t)\,dt = \sin(3t)\Big]_a^x = \sin(3x) - \sin(3a) = \sin(3x) - 1.$

Thus $\sin(3a) = 1$, so $3a = \frac{\pi}{2}$, and $a = \frac{\pi}{6}$.

Solution II. Let $x = a$, then $\sin 3a - 1 = \int_a^a f(t)\,dt = 0.$

Thus $\sin 3a = 1$, so $3a = \frac{\pi}{2}$, and $a = \frac{\pi}{6}$.

---

12. D p. 111

$xy^2 = 20$

Differentiate implicitly:

$y^2\frac{dx}{dt} + 2xy\frac{dy}{dt} = 0$

Since x is <u>decreasing</u> at a rate of 3 units/sec, we know that $\frac{dx}{dt} = -3$.

Also, when $y = 2$, then $x = 5$.

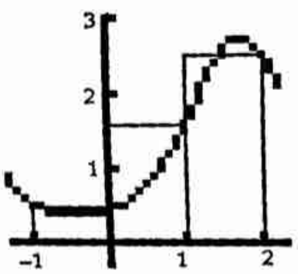
Substituting these values in the derivative equation yields

$4(-3) + 2(5)(2)\frac{dy}{dt} = 0$

Hence $20\frac{dy}{dt} = 12$ so $\frac{dy}{dt} = \frac{3}{5}$ units/sec.

---

13. C p. 111



$f(x) = e^{\sin(1.5x - 1)}$

We need a right-hand Riemann sum with three subintervals on the interval $[-1, 2]$. The subintervals to be dealt with are: $[-1, 0]$, $[0, 1]$, and $[1, 2]$.

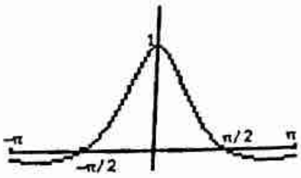
The right-hand function values are then:

$f(0) = 0.4311$
$f(1) = 1.6151$
$f(2) = 2.4826$

The sum of the area of the right-hand rectangles is then the sum of these three right-hand values (since each rectangle has width 1).

$R_3 = 0.4311 + 1.6151 + 2.4826 = 4.5288$

14. E p. 112



The graph of $\frac{dy}{dx} = \frac{\cos x}{x^2+1}$ shows two zeros (extrema for y) and three extrema (inflection points for y).

15. D p. 112

By the Second Fundamental Theorem, $G'(x) = f(x)$. Hence the given graph is the graph of the derivative of G.

I.	$G'(x) = f(x) = < 0$ on $(1,2)$, so G is <u>decreasing</u> there.	False
II.	$G'(x) = f(x) = < 0$ on $(-4,-3)$, so G <u>is</u> decreasing there.	True
III.	$G(-4) = 0$ (since it is the integral from -4 to -4). $G(x)$ is a decreasing function on the interval $[-4,0]$ (since its derivative is negative-valued there). Thus $G(0) < G(-4) = 0$	True

16. D p. 113

$\lim_{x\to1^-} f(x) = 1 + k - 3$ while $\lim_{x\to1^+} f(x) = 3 + b$.

For the function f to be continuous, these limits must be equal.

Hence $1+k-3 = 3+b$; this simplifies to $k = 5+b$.

For the function f to be differentiable, $\lim_{x\to1^-} f'(x)$ must equal $\lim_{x\to1^+} f'(x)$.

We find $\lim_{x\to1^-} f'(x) = \lim_{x\to1^-} (2x+k) = 2+k$.

also $\lim_{x\to1^+} f'(x) = \lim_{x\to1^+} (3) = 3$.

Setting these last two limits equal, we have $2+k = 3$, so $k = 1$.

Then from the previous condition ($k = 5+b$), we have $b = -4$.

17. B p. 113

The position function is an antiderivative of the velocity function:

$v(t) = t + 2\sin t$.

Hence $x(t) = \frac{1}{2}t^2 - 2\cos t + C$.

The given condition that $x(0) = 0$ implies $0 = 0 - 2\cos 0 + C$.

Therefore $C = 2$, and the position function is $x(t) = \frac{1}{2}t^2 - 2\cos t + 2$.

Now we must determine the moment when the velocity is 6.

$v(t) = 6 \quad \Rightarrow \quad t + 2\sin t = 6$.

We solve this equation with our calculator: $t = 6.1887$.

The position at that moment is $x(6.1886965) \approx 19.1589$.

Exam V
Section II
Part A — Calculators Permitted

1. p. 115

(a) The average velocity over the interval $1 \le t \le 3$ is given by

$$\frac{x(3)-x(1)}{3-1} = \frac{(e^3-\sqrt{3})-(e^1-\sqrt{1})}{2} = 8.318 \text{ ft/sec.}$$

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

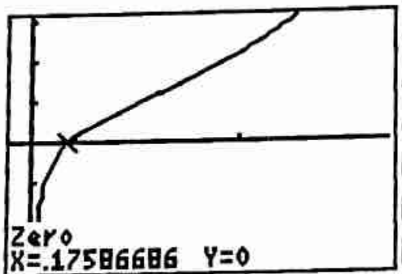
(b) The velocity function is given by

$$v(t) = x'(t) = e^t - \frac{1}{2\sqrt{t}}.$$

The velocity at time $t = 1$ is $v(1) = e - \frac{1}{2} = 2.218$ ft/sec.
Since the velocity at time $t = 1$ is positive, the particle is moving to the right (in the positive direction) on the x-axis at a speed of 2.218 ft/sec.

$3: \begin{cases} 1: \text{velocity } x'(t) \\ 2: \text{answer} \\ \quad 1: \text{direction} \\ \quad 1: \text{speed} \end{cases}$

(c) The particle is moving to the right when $v(t) = e^t - \frac{1}{2\sqrt{t}} > 0$. Shown to the right is a graph of the velocity function $v(t)$. When $t > 0.176$, the velocity is positive-valued. Hence the particle is moving to the right when $t > 0.176$ seconds.



$-.1 \le x \le 1.7$ $\quad -3.1 \le y \ge 3.1$

$2: \begin{cases} 1: \text{velocity } x'(t) > 0 \\ 1: \text{answer} \end{cases}$

(d) The velocity is 0 when $t = 0.176$ sec.
The particle's position at that moment is $x(0.176) = 0.773$ ft.

$2: \begin{cases} 1: \text{find } t \text{ when } x'(t) = 0 \\ 1: \text{answer} \end{cases}$

2. p. 116

(a)

$$\int_0^8 4e^{\sin(\frac{\pi}{12}t)}\,dt = 68.236 \text{ gallons}$$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) On the interval $0 \le t \le 8$, $w'(t) = \frac{\pi}{3} \cdot e^{\sin(\frac{\pi}{12}t)} \cdot \cos(\frac{\pi}{12}t)$ is equal to 0 and changes sign from positive to negative only once, when
$\frac{\pi}{12}t = \frac{\pi}{2} \Rightarrow t = 6$ hours

$3: \begin{cases} 1: \text{consider } w'(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c)

$$150 + \int_0^8 4e^{\sin(\frac{\pi}{12}t)}\,dt + \int_8^{12}\left[4e^{\sin(\frac{\pi}{12}t)} - \frac{13t}{1+2t}\right]dt = 220.116$$

$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(d)

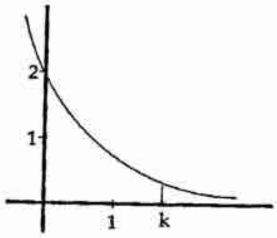
Amount of water is decreasing since $W(12) - R(12) = -2.24 < 0$

1: answer with reason

Exam V
Section II
Part B — No Calculators

3. p. 117

(a)



$$A = \int_0^k 2e^{-x}\,dx = \left[-2e^{-x}\right]_0^k = -2e^{-k} + 2 = 2 - \frac{2}{e^k}$$

3: {1: limits, 1: integrand, 1: answer}

(b) The volume of the solid revolved about the x-axis is done using the disk method.

$$V_x = \pi\int_0^k (2e^{-x})^2\,dx = \pi\int_0^k (4e^{-2x})\,dx = \left[-2\pi e^{-2x}\right]_0^k = -2\pi(e^{-2k} - 1) = 2\pi\left(1 - \frac{1}{e^{2k}}\right)$$

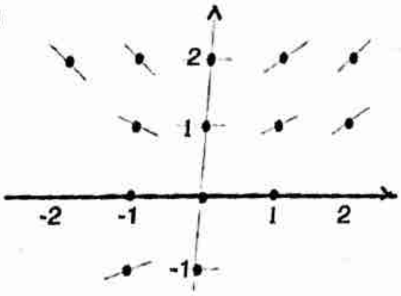
3: {1: limits, 1: integrand, 1: answer}

(c)
$$\lim_{k\to\infty} V_x = \lim_{k\to\infty} 2\pi\left(1 - \frac{1}{e^{2k}}\right) = 2\pi - 2\pi\left[\lim_{k\to\infty}\frac{1}{e^{2k}}\right] = 2\pi - 2\pi\cdot 0 = 2\pi$$

3: {1: limit, 2: answer}

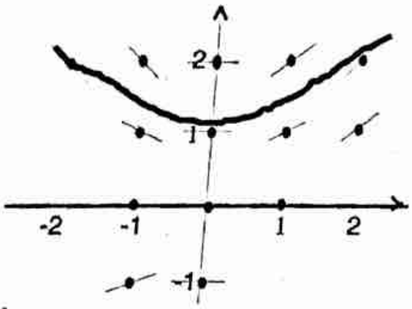
4. p. 118

(a)



2: { 1: zero slopes; 1: nonzero slopes }

(b)



1: solution curve mustgo through (-2,2), follow the given slope lines and extend to the border

(c) $\frac{dy}{dx} = \frac{xy}{x^2+4}$

We separate variables: $\frac{dy}{y} = \frac{x\,dx}{x^2+4}$.

Integrating both sides, we have $\ln|y| = \frac{1}{2}\ln|x^2+4| + C$.

This can be rewritten $\ln|y| = \ln\sqrt{x^2+4} + C$.

Taking exponentials of both sides yields

$$|y| = e^{\ln\sqrt{x^2+4} + C} = e^C \cdot \sqrt{x^2+4}.$$

This gives $y = D\sqrt{x^2+4}$, where $D = \pm e^C$, depending upon the initial conditions on y.

4: { 1: separate variables; 1: antiderivatives; 1: constant of integration; 1: solve for y }

(d) Substituting the point (0,4) into the general solution, we have

$4 = D\sqrt{0+4}$

From this we find that $D = 2$.

Hence the particular solution is $y = 2\sqrt{x^2+4}$.

2: { 1: solves for constant; 1: answer }

5. p. 119

(a) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 5 + 1\cdot(3+2) = 5 + 5 = 10$

1: answer

(b) f' changes from increasing to decreasing or vice versa at $x = -3$, 1 and 3.
Thus the graph of f has points of inflection at $x = -3$, 1 and 3.

2: { 1: $x = -3$, 1 and 3; 1: justification }

(c) f is an increasing function $\Leftrightarrow f'(x) > 0$.
This occurs on the intervals $(-6,-4)$ and $(-2, 6)$.
The graph of f is concave down where the graph of f' is decreasing.
This occurs on the intervals $(-6,-3)$ and $(1, 3)$.
Thus, the graph of f is increasing and concave down on the intervals $(-6,-4)$ and $(1, 3)$.

3: { 1: increasing intervals $(-6,-4)$, $(-2, 6)$; 1: concave down $(-6,-3)$, $(1, 3)$; 1: answer }

(d) $H'(x) = f'[g(x)]\cdot g'(x)$
At $x = 4$,
$H'(4) = f'[g(4)]\cdot g'(4)$.
With $g(x) = x^2 - 3x - 1$, $g(4) = 3$ and $g'(4) = 5$, we have
$H'(4) = f'[3]\cdot 5 = 2\cdot 5 = 10$

3: { 1: $H'(x) = f'[g(x)]\cdot g'$; 1: $g(4)$, $g'(4)$; 1: answer }

6. p. 120

(a) $f(x) = \ln\left[\frac{x}{x+1}\right]$

Since the domain of the natural logarithm function is the set of positive numbers, we must have $\frac{x}{x+1} > 0$.

This is true if and only if either $x > 0$ or $x < -1$.

Hence the domain of $f = \{\, x \mid x < -1 \text{ or } x > 0 \,\}$.

2: $\begin{cases} 1: \frac{x}{x+1} > 0 \\ 1: \text{answer} \end{cases}$

(b) Since $f(x)$ can be written $f(x) = \ln x - \ln(x+1)$ for $x > 0$,
and $f(x) = \ln|x| - \ln|x+1|$ for $x < -1$, we have in either case

$$f'(x) = \frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}.$$

2: answer

(c) $f(1) = \ln\frac{1}{2} = -\ln 2$ while $f'(1) = \frac{1}{2}$.

Thus the tangent line at the point $(1, f(1))$ has the equation

$$y + \ln 2 = \frac{1}{2}(x-1).$$

2: $\begin{cases} 1: \text{slope} \\ 1: \text{tangent equation} \end{cases}$

(d) To find an expression for $g(x)$, where g is the inverse of f, we interchange x and y in the rule for $f(x)$, and solve for y.

$$x = \ln\left[\frac{y}{y+1}\right] \Rightarrow e^x = \frac{y}{y+1}$$
$$\Rightarrow y e^x + e^x = y$$
$$\Rightarrow e^x = y - y e^x$$
$$\Rightarrow y(1-e^x) = e^x$$
$$\Rightarrow y = \frac{e^x}{1-e^x}$$

3: $\begin{cases} 1: \text{interchange } x \text{ and } y \\ 1: \text{solve for } y \\ 1: \text{answer} \end{cases}$

Thus $g(x) = f^{-1}(x) = \frac{e^x}{1-e^x}$.

Then $g'(x) = \frac{(1-e^x)\cdot e^x - e^x\cdot(-e^x)}{(1-e^x)^2} = \frac{e^x}{(1-e^x)^2}$

Exam VI
Section I
Part A — No Calculators

1. A p. 121

Using the product rule we obtain the first and second derivatives of $y = xe^x$.

$$y' = e^x + xe^x$$
$$y'' = e^x + e^x + xe^x$$
$$= 2e^x + 2e^x$$
$$= e^x(2 + x)$$

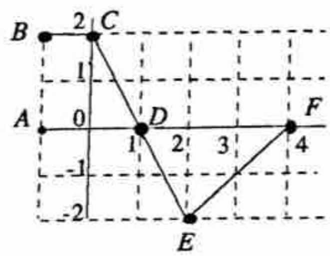
Thus y'' changes sign at $x = -2$.

2. C p. 121

Since $H(4) = \int_1^4 f(t)\,dt$, then

$H(4) =$ area trapezoid ABCD − area triangle DEF

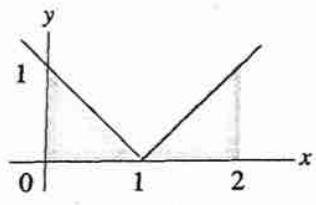
$$= \frac{1}{2}(2)(1+2) - \frac{1}{2}(3)(2) = 3 - 3$$
$$= 0$$



3. B p. 122

Solution I.

$$\int_0^2 |x-1|\,dx = \text{area of 2 equal triangles}$$
$$= 2\left(\frac{1}{2}\right)(1)(1) = 1.$$



Solution II.

$$\int_0^2 |x-1|\,dx = \int_0^1 -x+1\,dx + \int_1^2 x-1\,dx = -\frac{x^2}{2} + x\Big|_0^1 + \frac{x^2}{2} - x\Big|_1^2 = \frac{1}{2} + \frac{1}{2} = 1$$

4. E p. 122

(A), (B), and (D) are part of the definition of continuity and hence true. (C) is a statement equivalent to (A). (E) is the only one that could be false, for example, consider $y = |x|$ at $x = 0$.

5. C p. 122

$$\int_0^x 2\sec^2\left(2t+\frac{\pi}{4}\right)dt = \tan\left(2t+\frac{\pi}{4}\right)\Big|_0^x = \tan\left(2x+\frac{\pi}{4}\right) - \tan\frac{\pi}{4} = \tan\left(2x+\frac{\pi}{4}\right) - 1.$$

---

6. A p. 123

Using implicit differentiation on $xy + x^2 = 6$ we obtain

$$y + xy' + 2x = 0 \qquad \text{At } x = 1$$
$$xy' = -2x - y \qquad -y + 1 = 6 \text{ and } y = -5$$
$$y' = \frac{-2x - y}{x} \qquad \text{Thus } y' = \frac{2+5}{-1} = -7$$

---

7. C p. 123

Solution I.

$$\int_2^3 \frac{x}{x^2+1}dx = \frac{1}{2}\int_2^3 \frac{1}{x^2+1}(2x\,dx) = \frac{1}{2}\ln(x^2+1)\Big|_2^3 = \frac{1}{2}[\ln 10 - \ln 5] = \frac{1}{2}\ln 2$$

Solution II.

Let $u = x^2 + 1$, then $du = 2x\,dx$. Then

$$\int_2^3 \frac{x}{x^2+1}dx = \int_5^{10} \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}\ln u\Big|_5^{10} = \frac{1}{2}(\ln 10 - \ln 5) = \frac{1}{2}\ln 2.$$

---

8. E p. 123

$f'(x) = e^{\sin x}$

I. $f''(x) = e^{\sin x}\cos x$ $\quad f''(0) = 1\cdot 1 = 1$ TRUE

II. Slope of $y = x + 1$ is 1, and the line goes through (0, 1) TRUE

III. $h(x) = f(x^3 - 1)$ $\quad h'(x) = f'(x^3-1)\cdot(3x^2) = 3x^2e^{\sin(x^3-1)}$

Since $h'(x) \ge 0$ for all x, then the graph of $h(x)$ is increasing. TRUE

---

9. D p. 124

The total flow is $\int_0^6 f(t)\,dt$ which is equal to the area under the curve. Each square under the curve represents 10 gallons. There are 11 full squares and 2.5 part squares. Hence, the approximate area is $13.5 \times 10 = 135$ gallons.

---

10. B p. 124

The instantaneous rate of change of $f(x) = e^{2x} - 3\sin x$ is

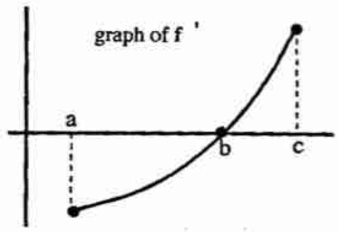
$$f'(x) = 2e^{2x} - 3\cos x.$$

Thus, $f'(0) = 2e^0 - 3\cos 0 = 2 - 3 = -1.$

---

11. E p. 124

f' is negative on (a, b) so f is decreasing there and (A) is false. f' increases on (a, c), hence, f'' is positive and the graph of f is concave up. As a result, (B), (C), and (D) are also false.
(E) is true because f is an antiderivative of f'.



---

12. E p. 125

Using the Fundamental Theorem and the Chain Rule on

$F(x) = \int_0^{x^2} \frac{1}{2+t^3}\,dt$, we obtain $F'(x) = \frac{1}{2+(x^2)^3} \cdot 2x.$

Then, $F'(-1) = \frac{1}{2+1} \cdot (-2) = -\frac{2}{3}.$

---

13. D p. 125

f is not differentiable at $x = 0$ because $\lim_{x \to 0^-} f'(x) \neq \lim_{x \to 0^+} f'(x).$

f is not differentiable at $x = 3$ because f is not continuous there.

Hence the answer is (D).

---

14. D p. 125

The velocity $v(t) = x'(t) = \frac{(t^2+4)(1) - t(2t)}{\left(t^2+4\right)^2} = \frac{4-t^2}{\left(t^2+4\right)^2} = \frac{(2+t)(2-t)}{\left(t^2+4\right)^2}.$

For $t \geq 0$, $v(t) = 0$ when $t = 2$.

15. D p. 126

To find the maximum value of $f(x) = 2x^3 + 3x^2 - 12x + 4$ on the closed interval [0,2], we evaluate the function f at the critical numbers and endpoints and take the largest resulting value.

$f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$.

Thus in the closed interval [0,2], the only critical number is $x = 1$. The values of the function at the one critical number and at the endpoints are

critical value: $f(1) = -3$
endpoints: $f(0) = 4$ and $f(2) = 8$

From this list we see that the maximum value is 8.

---

16. A p. 126

Using the Chain Rule twice on $f(x) = \ln(\cos 2x)$, we obtain

$$f'(x) = \frac{1}{\cos 2x} \cdot (-\sin 2x) \cdot (2).$$

$$= -2\frac{\sin 2x}{\cos 2x} = -2\tan 2x.$$

---

17. A p. 126

The slopes of the segments in the slope field vary as x changes. Thus the differential equation is not of the form $\frac{dy}{dx} = g(y)$. That eliminates B, C and E of the five choices. If the differential equation were $\frac{dy}{dx} = x^2 + y^2$, then slopes would always be at least 0. They aren't in Quadrant III. That eliminates choice (D).

---

18. E p. 127

$$y = \sqrt{x+3} = (x+3)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(x+3)^{-\frac{1}{2}}$$

At (1,2), $y' = \frac{1}{2}\frac{1}{\sqrt{1+3}} = \frac{1}{4}$ and the equation of the line is $y - 2 = \frac{1}{4}(x-1)$.

When $x = 0$, then $y - 2 = \frac{1}{4}(0-1)$ and $y = \frac{7}{4}$.

---

19. B p. 127

Solution I.

Using the Product Rule twice on $f(x) = (x-1)(x+2)^2$ gives

$$f'(x) = (x+2)^2 + (x-1)\cdot 2(x+2)$$

$$f''(x) = 2(x+2) + 2(x+2) + 2(x-1)$$
$$= 6x + 6$$

Thus, $f''(x) = 0$ when $x = -1$.

Solution II.

Multiplying out $f(x) = (x-1)(x+2)^2$ gives

$$f(x) = (x-1)(x+2)^2$$
$$= (x-1)(x^2+4x+4) = x^3 + 3x^2 - 4$$
$$f'(x) = 3x^2 + 6x$$
$$f''(x) = 6x + 6, \text{ thus } f''(x) = 0 \text{ when } x = -1.$$

---

20. C p. 127

$$\int_0^b (4bx - x^2)\,dx = \frac{4bx^2}{2} - \frac{2x^3}{3}\Big|_0^b = 2b^3 - \frac{2b^3}{3} = \frac{4b^3}{3}$$

Thus $\frac{4b^3}{3} = 36 \Rightarrow b^3 = 27$ and $b = 3$.

---

21. C p. 128

Separating variables for $\frac{dy}{dx} = \frac{1}{3}y^{-2}$, gives: $3y^2 dy = 1dx$.

Integrating gives $y^3 = x + C$ At $(-1,-1)$, $-1 = -1 + C \Rightarrow C = 0$.

Thus, $y^3 = x \Rightarrow y = x^{1/3}$

The largest domain that contains $(-1,-1)$ is $x < 0$.

Thus, $y = x^{1/3}$ $x < 0$..

---

22. C p. 128

$f(x) = x^3 - 5x^2 + 3x$

$f'(x) = 3x^2 - 10x + 3 = (3x - 1)(x - 3)$ and $f'(x) < 0$ when $\frac{1}{3} < x < 3$.

$f''(x) = 6x - 10$ and $f''(x) < 0$ when $x < \frac{5}{3}$.

Thus $\frac{1}{3} < x < \frac{5}{3}$ satisfies both.

---

23. C p. 128

I is false as $f'(x) < 0$ on $(1, 3)$, hence f is decreasing on part of $(2, 4)$.

II is false as $f'(x)$ goes from plus to minus at $x = 2$, hence f has a relative maximum there.

III is true as f' has a horizontal tangent there which means that $f'(x) = 0$ and since the slope of f' changes sign there is an inflection point at $x = 1$.
Thus the answer is C.

---

24. B p. 129

$f(x) = \arctan(2x - x^2)$

$f'(x) = \dfrac{2 - 2x}{1 + (2x - x^2)^2}$

$f'(x) = 0$ when $x = 1$.

$1 + (2x - x^2)^2 > 0$, so f' exists for all $x \in \Re$.

Thus $x = 1$ is the only critical value.

---

25. C p. 129

I is true because $\lim_{x \to 1} |x - 1| = 0$ and $f(1) = 0$, so f is continuous at $x = 1$.

II is true because $y = e^x$ is continuous for all real x and $y = x - 1$ is continuous for all x, hence the composite is continuous for all x.

III is false because at $x = 1$, $\ln(e^0 - 1) = \ln(1 - 1) = \ln 0$ which is undefined.

Thus the answer is (C).

26. D p. 129
The number of motels is

$$\int_0^5 m(x)\,dx = \int_0^5 11 - e^{0.2x}\,dx = 11x - \frac{1}{0.2}e^{0.2x}\Big|_0^5 = 11x - 5e^{0.2x}\Big|_0^5$$

$$= (55 - 5e^1) - (0 - 5e^0)$$

$$= 55 - 5e + 5$$

$$= 60 - 5e$$

Since $e \approx 2.71$ and $5e \approx 13.55$, the number of motels is approximately $60 - 13.55 \approx 46$.

(*N.B.* Use $e \approx 3 \Rightarrow 5e \approx 15 \Rightarrow 60 - 5e \approx 45$ and (D) is still the best approximation.)

---

27. C p. 130

The relationship between x, y, and z is Pythagorean: $z^2 = x^2 + y^2$.
The derivative with respect to time, t, gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \quad (*)$$

When $x = 3$ and $y = 4$, then $z = 5$.

Substituting $3\frac{dx}{dt}$ for $\frac{dy}{dt}$ and 2 for $\frac{dz}{dt}$, in (*) we obtain

$$(5)(2) = (3)\frac{dx}{dt} + 4\left(3\frac{dx}{dt}\right)$$

$$10 = 15\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{2}{3}.$$

---

28. E p. 130

$$f(x) = \sin(2x) + \ln(x+1)$$

$$f'(x) = \cos(2x)\cdot 2 + \frac{1}{x+1}$$

$$f'(0) = \cos(0)\cdot 2 + \frac{1}{0+1} = 2 + 1 = 3.$$

nExam VI
Section I
Part B — Calculators Permitted

1. E p. 131

(A) is true because in the neighborhood of $x = a$, $f(a)$ is the smallest value.
(B) and (C) and (D) are true from reading the graph.
(E) is false because f' does not exist at $x = a$. (Use the contrapositive of the theorem: If $f'(a)$ exists, then f is continuous at $x = a$; that is, if f is not continuous at $x = a$, then $f'(a)$ does not exist.)

2. C p. 131

A horizontal tangent occurs when $f'(x) = 0$.

$$f(x) = e^{3x} + 6x^2 + 1$$

$$f'(x) = 3e^{3x} + 12x$$

Thus $f'(x) = 0$ when $x = -0.156$.

3. C p. 132

Differentiating $PV = c$ with respect to t gives $\frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = 0$.

At $P = 100$ and $V = 20$ and $\frac{dV}{dt} = -10$, we have $\frac{dP}{dt} \cdot 20 + 100(-10) = 0$

$$\frac{dP}{dt} = \frac{1000}{20} = 50 \frac{\text{lb/in}^2}{\text{sec}}$$

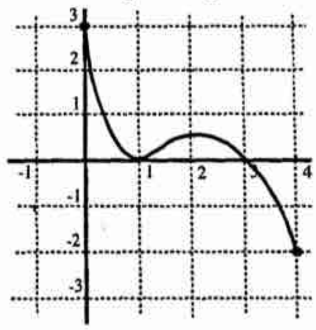
4. C p. 132

I is false because g'' changes sign only at $x = 3$.

II is true because $g''(x) < 0$ on $(3, 4)$.

III is true because $g''(x) \geq 0$ on $(0, 2)$, thus $g'(x)$ is non-decreasing there. Since $g'(0) = 0$ and g' does not decrease, then $g'(x) > 0$ at $x = 2$ and g is increasing there. Thus, the answer is C.

Graph of g''



5. E p. 132

To find the maximum velocity on the closed interval [0, 3], we evaluate the velocity function at the critical numbers and endpoints and take the largest resulting value.

Using the Fundamental Theorem to find the derivative of $s(t) = \int_0^t (x^3 - 2x^2 + x)\,dx$ gives

$v(t) = s'(t) = t^3 - 2t^2 + t$

$v'(t) = 3t^2 - 4t + 1 = (3t - 1)(t - 1)$

Thus on the closed interval [0, 3], the critical number are $t = 1$ and $t = \frac{1}{3}$.
The values of the velocity function at the critical numbers and at the endpoints are

critical value: $v\left(\frac{1}{3}\right) = \frac{4}{27}$ and $v(1) = 0$
endpoints: $v(0) = 0$ and $v(3) = 12$

From this list we see that the maximum velocity is 12 m/sec

---

6. D p. 133

Solution I. The rate of change $= \frac{dy}{dx} = \sqrt{2x+1}$.

On [0, 4], the average rate of change $= \frac{1}{4-0}\int_0^4 \sqrt{2x+1}\,dx$

$$= \frac{1}{4}\int_0^4 (2x+1)^{1/2}\,dx = \frac{1}{4}\cdot\left[\frac{1}{2}(2x+1)^{3/2}\cdot\frac{2}{3}\right]$$

$$= \frac{1}{12}\left[9^{3/2} - 1\right] = \frac{1}{12}[27 - 1] = \frac{13}{6}.$$

Solution II. Average rate of change $= \frac{1}{4}\,\text{fnInt}\left(\sqrt{2x+1}, x, 0, 4\right) = 2.1666$

---

7. A p. 133

Since $f(x)$ is an antiderivative of $f'(x)$, then $\int_0^2 f'(x)\,dx = f(2) - f(0) = 0$.

---

8. B p. 134

$$\lim_{x\to k} \frac{x^2 - k^2}{x^2 - kx} = \lim_{x\to k} \frac{(x+k)(x-k)}{x(x-k)} = \lim_{x\to k} \frac{(x+k)}{x} = 2$$

9. B p. 134

Let w = the weight of the duck, then $\frac{dw}{dt} = kw$.

Separating the variables gives $\frac{1}{w}dw = k\,dt$.

Integrating yields $\ln w = kt + c$ When $t = 0$ and $w = 2$, then $C = 2$,

$w = e^{kt+c} = Ce^{kt}$. when $t = 4$ and $w = 3.5$, then $3.5 = 2e^{4k}$.

Thus $4k = \ln 3.5$ and $k = 0.140$. At $t = 6$, $w = 2e^{0.14(6)} = 4.63$.

---

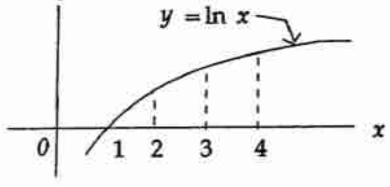
10. D p. 135

$\Delta x = \frac{4-1}{3} = 1 \quad x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4$.

$T_3 = \frac{1}{2}(\Delta x)[y_0 + 2y_1 + 2y_2 + y_3]$

$= \frac{1}{2}(1)[0 + 2\ln 2 + 2\ln 3 + \ln 4]$

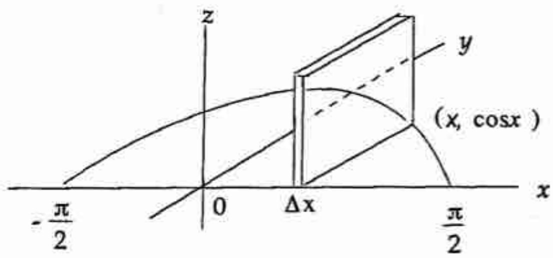
$= 2.4849$



---

11. C p. 135

$V = 2\int_0^{\pi/2} (\cos x)^2\,dx = 1.5708$



---

12. E p. 136

Consider points $P = (a-h, f(a-h))$, $Q = (a, f(a))$, and $R = (a+h, f(a+h))$.
The definition of the derivative of f at $x = a$ is the limit of the slope of the secant.

For $\overline{PQ}$ $f'(a) = \lim_{h\to 0}\frac{f(a)-f(a-h)}{a-(a-h)} = \lim_{h\to 0}\frac{f(a)-f(a-h)}{h}$. II

For $\overline{QR}$ $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{a+h-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$. I

For $\overline{PR}$ $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a-h)}{(a+h)-(a-h)} = \lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}$. III

Since f is differentiable, all the limits exist and all three are true.

13. A p. 136

$$y_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Using fnInt or graphing and using ∫f(x) dx for $L(x) = e^{-0.1x} + \frac{1}{x^2}$ on $a = 15$ to $b = 25$

gives $L_{ave} = \frac{1}{25-15}\int_{15}^{25} e^{-0.1x} + \frac{1}{x^2}\,dx = \frac{1}{10}(1.437) = 0.144.$

---

14. D p. 137

The graph of $y = f'(x) = e^x\sin(x^2) - 1$ has three zeros on [0, 3]. $x = 0.715$ is a relative minimum, $x = 1.721$ is a relative maximum, and $x = 2.523$ is a relative minimum. Thus I and II are true.

The graph of f' has three horizontal tangents on [0, 3], so f has 3 inflection points and III is false.

---

15. B p. 137

$$y_{ave} = \frac{1}{b-1}\int_1^b x^2\,dx = \frac{1}{b-1}\left.\frac{x^3}{3}\right|_1^b = \frac{1}{3(b-1)}(b^3 - 1).$$

Solution I. Using the calculator to find the zeros of

$\frac{1}{3(b-1)}(b^3 - 1) = \frac{13}{3}$, we obtain $b = 3$.

Solution II. $\frac{1}{3(b-1)}(b^3 - 1) = \frac{(b-1)(b^2+b+1)}{3(b-1)} = \frac{b^2+b+1}{3}$.

Solving $\frac{b^2+b+1}{3} = \frac{13}{3}$. gives $b^2 + b + 1 = 13$,

then $b^2 + b + 12 = 0$, and $(b+4)(b-3) = 0$, so $b = -4$ or $b = 3$.

The only solution on $[1, b]$ is $b = 3$.

16. C p. 138

$$f(x)=\frac{2x}{x^2+1}$$

Substituting $u=x^2+1$ and $du=2x\,dx$ with $x=0\Rightarrow u=1$ and

$x=3\Rightarrow u=10$ into $\int_0^3 f(x)\,dx$ gives $\int_1^{10}\frac{1}{u}\,du=\ln u\Big|_1^{10}=\ln 10$.

Thus I is true.

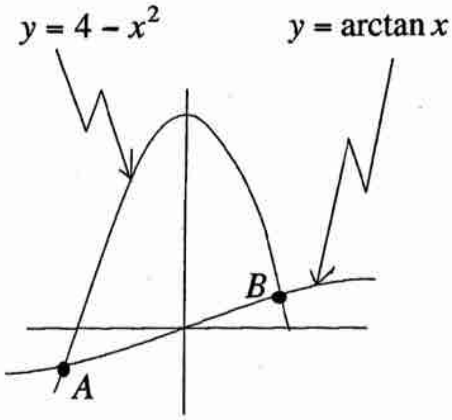
$$f'(x)=\frac{(x^2+1)(2)-2x(2x)}{(x^2+1)^2}=\frac{2-2x^2}{(x^2+1)^2}=\frac{2(1-x)(1+x)}{(x^2+1)^2}.$$

From f' we see the critical values are ±1. At $x=1$, $f'(x)$ goes from positive to negative, so (1, 1) is a relative maximum and II is true.

$$f'(2)=\frac{2(1-2)(1+2)}{(2^2+1)^2}=-\frac{6}{25},$$ thus III is false.

17. B p. 138

Using the graphing calculator, we graph $y=\arctan x$ and $y=4-x^2$ on the same axes. Next determine the intersection points $A=(-2.270,-1.156)$ and $B=(1.719, 1.044)$.



The area under $y=4-x^2$ and above $y=\arctan x$ is given by

$$\int_{-2.270}^{1.719}[(4-x^2)-\arctan x]\,dx=10.972.$$

Exam VI
Section II
Part A — Calculators Permitted

1 p. 140

(a) Divide [0, 8] into four equal subintervals of [0, 2], [2, 4], [4, 6], and [6, 8] with midpoints $m_1 = 1$, $m_2 = 3$, $m_3 = 5$, and $m_4 = 7$.

The Riemann sum is $S_4 = \sum_{i=1}^{4} \Delta t \cdot R(m_i)$

where $R(1) = 5.4$, $R(3) = 6.5$, $R(5) = 6.3$, and $R(7) = 5.5$.

$S_4 = 2(5.4) + 2(6.5) + 2(6.3) + 2(5.5) = 47.4$ metric tons pumped out.

3: { 1: $R(1) + R(3) + R(5) + R(7)$; 1: answer; 1: meaning with units }

(b) Since R(2) = R(6) = 6.1, the Mean Value Theorem guarantees a time t between t = 2 and t = 6 when $R'(t) = \frac{R(6) - R(2)}{6 - 2} = 0$.

3: { 1: yes; 2: MVT or equivalent }

(c) $Q_{ave} = \frac{1}{8-0}\int_0^8 \frac{1}{8}(36 + 8t - t^2)\,dt$

$= \frac{1}{64}\left(36t + 4t^2 - \frac{t^3}{3}\right]_0^8 = 5.833$ metric tons per hour.

3: { 1: limits; 1: integrand; 1: answer }

2. p. 141

(a) To find the maximum velocity we first determine the critical numbers.

The derivative $v'(t) = \dfrac{(1+t^2) - t(2t)}{(1+t^2)^2} = \dfrac{1-t^2}{(1+t^2)^2} = \dfrac{(1+t)(1-t)}{(1+t^2)^2}$ exists and is continuous everywhere. Thus, the only critical numbers are inputs for which $v'(t) = 0$. From the factorization of v' it is clear that $v'(t) = 0 \Rightarrow t = \pm 1$.

If $t < 1$, then $v'(t) > 0$ and f is increasing; if $t > 1$, then $v'(t) < 0$ and f is decreasing. Thus, f attains a relative maximum at $t = 1$. In fact, since f always increases to the left of $t = 1$ and always decreases to the right of $t = 1$, it is clear that f attains its maximum value at $t = 1$ at which its maximum velocity is

$$v(1) = 1 + \frac{1}{1+t^2} = \frac{3}{2} = 1.5\,.$$

3: 1: Sets $v'(t)=0$; 1: Identifies $x = 1$ as a candidate; 1: answer and explanation

(b) The position is found by integrating the velocity function.

$$s(t) = \int 1 + \frac{t}{1+t^2}\,dt = t + \frac{1}{2}\ln(1+t^2) + C.$$

At time $t = 0$, the particle is at $x = 1$, thus

$$1 = 0 + \frac{1}{2}\ln(1) + C \text{ and } C = 1\,.$$

Hence, $s(t) = t + \frac{1}{2}\ln(1+t^2) + 1$

3: 1: $\int v(t)\,dt$; 1: antiderivative; 1: answer

(c) $$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(1 + \frac{\frac{1}{t}}{\frac{1}{t^2}+1}\right) = 1 + \frac{0}{0+1} = 1$$

1 : answer

(d) The position at time t is $s(t) = t + \frac{1}{2}\ln(1+t^2) + 1$.

Solving $t + \frac{1}{2}\ln(1+t^2) + 1 = 101$ or $t + \frac{1}{2}\ln(1+t^2) + 1 - 100 = 0$ using the graphing calculator gives $t = 95.441$.

2: 1: solves $s(t) = 101$; 1: answer

Exam VI
Section II
Part B — No Calculators

3. p. 142

(a) Area $= \int_1^8 \frac{8}{\sqrt[3]{x}}\,dx = \int_1^8 8x^{-1/3}\,dx = 8x^{2/3}\cdot\frac{3}{2}$

$= 12(8)^{2/3} - 12(1)^{2/3} = 48 - 12 = 36.$

3: $\begin{cases} 1\text{: limits} \\ 1\text{: integrand} \\ 1\text{: answer} \end{cases}$

(b) The area from $x = 1$ to $x = k$ is $\int_1^k 8x^{-1/3}\,dx = 12k^{2/3} - 12$.

Solving $12k^{2/3} - 12 = \frac{5}{12}\cdot 36 = 15$ gives $k^{2/3} = \frac{15+12}{12} = \frac{9}{4}$

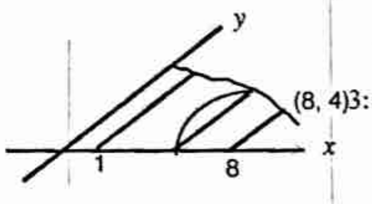
$k = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \frac{27}{8}.$

3: $\begin{cases} 2\text{: definite integral} \\ \quad 1\text{: limits} \\ \quad 1\text{: integrand} \\ 1\text{: answer} \end{cases}$

(c) Volume $= \int_1^8 \frac{\pi}{8} y^2\,dx = \int_1^8 \frac{\pi}{8}\left(\frac{8}{\sqrt[3]{x}}\right)^2 dx$

$= \int_1^8 8\pi x^{-2/3}\,dx = 8\pi\left(\frac{x^{1/3}}{1/3}\right)_1^8$

$= 24\pi\sqrt[3]{x}\Big|_1^8 = 24\pi(2-1) = 24\pi$



3: $\begin{cases} 1\text{: limits} \\ 1\text{: integrand} \\ 1\text{: answer} \end{cases}$

4. p. 143

(a) The acceleration is the rate of change of velocity; that is, the slope of the velocity graph. The rocket's fuel was expended at $t = 4$ at which point it continued upward but slowed down until $t = 7$ when it started to fall. Thus the acceleration of the rocket during the first 4 seconds is
$a = v' = \dfrac{96-0}{4-0} = 24$ ft per sec^2

3: {1: fuel expended at t = 4; 1: $v'(t)$ = line slope; 1: answer / units}

(b) The rocket goes up while v is positive. $v > 0$ for $0 < t < 7$ seconds. The rocket rises for 7 seconds.

2: {1: $v(t) > 0$; 1: answer}

(c) The distance traveled on the interval [0, 7] is

$\int_0^7 v(t)\,dt$ = the area of the triangle above the x-axis $= \frac{1}{2}(7)(96) = 336$ feet.

2: {1: triangle area $= \int_0^7 v(t)\,dt$; 1: answer}

(d) The height of the tower is the difference between the distance the rocket falls on $7 < t < 14$ and the distance it rises from part (c).

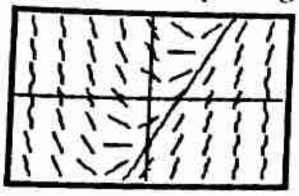
$\int_7^{14} v(t)\,dt$ = area of the triangle below the x-axis $= \frac{1}{2}(7)(224) = 784$ feet.

The height of the tower is $784 - 336 = 448$ feet.

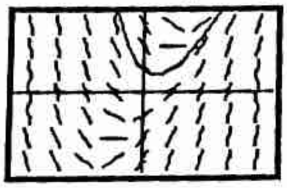
2: {1: triangle area; 1: answer}

5. p. 144

(a) solution curve passing through (1,0).



solution curve passing through (0,1)



$2: \begin{cases} 1\text{: curve through } (1,0) \\ 1\text{: curve through } (0,1) \end{cases}$

(b)

If $y = 2x + b$ is to be a solution, then $\frac{dy}{dx} = 2$. But we are given that $\frac{dy}{dx} = 2x - y$. Hence $2 = 2x - y$. Therefore $y = 2x - 2$ and $b = -2$.

1: answer

(c) At the point (0,0), $\frac{dy}{dx} = 2x - y = 0$. Therefore the graph of g has a horizontal tangent at (0,0). Differentiating $\frac{dy}{dx}$ implicitly, we have

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y)$$

At the point (0,0), this gives $\frac{d^2y}{dx^2} = 2$. That means that the graph of g is concave up at the origin. Since g has a horizontal tangent and is concave up at the origin, it has a local minimum there.

$4: \begin{cases} 1\text{: graph of g has a horizontal tangent at } (0,0) \\ 1\text{: show } g''(0) = 2 \\ 1\text{: answer} \\ 1\text{: justification} \end{cases}$

(d) The derivative of $y = Ce^{-x} + 2x - 2$ is $\frac{dy}{dx} = -Ce^{-x} + 2$. Substituting the given equation $\frac{dy}{dx} = 2x - y$ gives $2x - y = -Ce^{-x} + 2$. Solving for y we obtain $y = Ce^{-x} + 2x - 2$, which is the given solution.

$2: \begin{cases} 1\text{: } \frac{dy}{dx} \\ 1\text{: substitution} \end{cases}$

6. p. 145

(a)

$$G(x) = \int_{-3}^{x} f(t)\,dt \quad \text{and} \quad H(x) = \int_{2}^{x} f(t)\,dt$$

By the Second Fundamental Theorem, $G'(x) = f(x)$ and $H'(x) = f(x)$. Since the derivatives of G and H are the same, G and H differ by a constant. That is, $G(x) - H(x) = C$ for all x in the domain. This means that the graph of G is the same as the graph of H, moved C units vertically.

To evaluate the constant C, note that: $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$.

In particular, $\int_{-3}^{x} f(t)\,dt = \int_{-3}^{2} f(t)\,dt + \int_{2}^{x} f(t)\,dt$.

Thus $G(x) = \int_{-3}^{2} f(t)\,dt + H(x)$. Hence $G(x) - H(x) = \int_{-3}^{2} f(t)\,dt$. Using the

areas of the triangles $C = \int_{-3}^{2} f(t)\,dt = -\frac{1}{2}(2)(3) + \frac{1}{2}(1)(1) = -\frac{7}{2}$.

2: { 1: G(x) and H(x) differ by a constant; 1: geometric explanation }

(b)

H is increasing $\Leftrightarrow$ $H'(x) = f(x) > 0$.
This occurs on the intervals (-5,-3) and (1,5).

3: { 1: $H'(x) = f(x) > 0$; 2: answer }

(c) G has relative maximum values wherever H does. Because of the result of part (b), this will be at $x = -3$, where g'(x) changes sign from plus to minus.

2: { 1: answer; 1: justification }

(d) G is concave up $\Leftrightarrow$ $G'(x) = f(x)$ is increasing.
This occurs on the intervals (-2, 1) and (1, 3).
G"(1) doesn't exist because $f'(1^-) = \frac{2}{3}$ while $f'(1^+) = 1$.

2: { 1: $G'(t) = f(t)$ is increasing; 1: intervals }